CONVOLUTION ARRAYS FOR JACOBSTHAL AND FIBONACCI POLYNOMIALS

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The Jacobsthal polynomials and the Fibonacci polynomials are known to be related to Pascal's triangle and to generalized Fibonacci numbers [1]. Now, we show relationships to other convolution arrays, and in particular, we consider arrays formed from sequences arising from the Jacobsthal and Fibonacci polynomials, and convolutions of those sequences. We find infinite sequences of determinants as well as arrays of numerator polynomials for the generating functions of the columns of the arrays of Jacobsthal and Fibonacci number sequences, which are again related to the original Fibonacci numbers.

1. INTRODUCTION

The Jacobsthal polynomials $J_n(x)$,

(1.1) $$J_0(x) = 0, \quad J_1(x) = 1, \quad J_{n+2}(x) = J_{n+1}(x) + xJ_n(x),$$

and the Fibonacci polynomials $F_n(x)$,

(1.2) $$F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+2}(x) = xF_{n+1}(x) + F_n(x),$$

have both occurred in [1] as related to Pascal's triangle and convolution arrays for generalized Fibonacci numbers. We note that $F_n(1) = J_n(1) = F_n$, the nth Fibonacci number 1, 1, 2, 3, 5, 8, 13, ..., while $F_n(2) = P_n$, the nth Pell number 1, 2, 5, 12, 29, ... We list the first polynomials in these sequences below.

	$F_n(x)$	$J_n(x)$
$n = 1$	1	1
$n = 2$	x	1
$n = 3$	$x^2 + 1$	$1 + x$
$n = 4$	$x^3 + 2x$	$1 + 2x$
$n = 5$	$x^4 + 3x^2 + 1$	$1 + 3x + x^2$
$n = 6$	$x^5 + 4x^3 + 3x$	$1 + 4x + 3x^2$
$n = 7$	$x^6 + 5x^4 + 6x^2 + 1$	$1 + 5x + 6x^2 + x^3$
$n = 8$	$x^7 + 6x^5 + 10x^3 + 4x$	$1 + 6x + 10x^2 + 4x^3$
$n = 9$	$x^8 + 7x^6 + 15x^4 + 10x^2 + 1$	$1 + 7x + 15x^2 + 10x^3 + x^4$

Notice that the coefficients of $J_n(x)$ and $F_n(x)$ appear upon diagonals of Pascal's triangle, written as a rectangular array:

(1.3) $$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 2 & 3 & 4 & 5 & \ldots \\ 1 & 3 & 6 & 10 & 15 & \ldots \\ 1 & 4 & 10 & 20 & 35 & \ldots \\ 1 & 5 & 15 & 35 & 70 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

The diagonals considered are formed by starting from successive elements in the left-most column and progressing two elements up and one element right throughout the array. We shall call this the 2,1-diagonal, and we shall call such a diagonal formed by moving up p units and right q units the p,q-diagonal.

The sums of the elements on the p,q-diagonals of Pascal's triangle are the numbers $u(n;\, p-1,\, q)$ of Harris & Styles [4].

We shall display sequences of convolution arrays in what follows: If

$$\left\{a_n\right\}_{n=0}^{\infty} \quad \text{and} \quad \left\{b_n\right\}_{n=0}^{\infty}$$

are two sequences of integers, then their convolved sequence

$$\left\{c_n\right\}_{n=0}^{\infty}$$

is given by

$$c_0 = a_0 b_0, \quad c_1 = a_0 b_1 + b_0 a_1, \quad c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0,$$

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}. \tag{1.4}$$

Notice that this is the Cauchy product if a_n, b_n, c_n are coefficients of infinite series. The convolution array for a given sequence will contain the successive sequences formed by convolving a sequence with itself.

Pascal's triangle itself is the convolution array for powers of one. Looking back at the display (1.3), we find that the sums of elements appearing on the 1,1-diagonal are 1, 2, 4, ..., 2^n, ...; on the 2,1-diagonal are 1, 1, 2, 3, 5, ..., F_n, ...; on the 1,2-diagonal, 1, 2, 5, 13, ..., F_{2n-1}, ..., while the coefficients of $(1+x)^n$ appear on the 1,1-diagonal, and those of $F_n(x)$ and $J_n(x)$ appear on the 2,1-diagonal.

The convolution array for the powers of 2 is

$$\begin{array}{lcccccc} & 1 & 1 & 1 & 1 & 1 & \ldots \\ & 2 & 4 & 6 & 8 & 10 & \ldots \\ (1.5) & 4 & 12 & 24 & 40 & 60 & \ldots \\ & 8 & 32 & 80 & 160 & 280 & \ldots \\ & 16 & 80 & 240 & 560 & 1120 & \ldots \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

Notice that the sums of elements appearing on the 1,1-diagonal are powers of 3, and that the 1,1-diagonal contains coefficients of $(2+x)^n$. The 2,1-diagonals contain the coefficients of $F_{n+2}^*(x) = 2xF_{n+1}^*(x) + F_n^*(x)$, $F_1^*(x) = 1$, $F_2^*(x) = 2x$, and have the Pell numbers 1, 2, 5, 12, 29, ..., as sums, while the 1,2-diagonal sums are the sequence 1, 3, 11, 43, 171, ..., $J_{2n-1}(2)$, ... Noting that in the first array, $F_n = F_n(1)$, while in the second array the Pell numbers are given by $F_n(2)$, it would be no surprise to find that the numbers $F_n(3)$ appear as 2,1-diagonal sums in the powers of 3 convolution array. In fact,

Theorem 1.1: When the powers of k convolution array is written in rectangular form, the sums of elements appearing on the 1,1-diagonals are the powers of $(k+1)$, while the 1,1-diagonal contains the coefficients of $(k+x)^n$. The numbers given by $F_n(k)$ appear as successive sums of the elements of the 2,1-diagonals, which contain the coefficients of the polynomials $F_n^*(x)$, where

$$F_{n+2}^*(x) = kxF_{n+1}^*(x) + F_n^*(x),\ F_1^*(x) = 1,\ F_2^*(x) = kx.$$

The sums of the elements appearing on the 1,2-diagonal are given by the numbers $J_{2n-1}(x)$.

Proof: Since the powers of k are generated by $1/(1 - kx)$, the numbers $F_n(k)$ by $1/(1-kx-x^2)$, and $J_{2n-1}(x)$ by $(1-kx)/(1-(2k+1)x+k^2x^2)$, the results of Theorem 1.1 follow easily from Theorem 1.2 with proper algebra.

We need to write the generating function for the sums of elements appearing on the p,q-diagonal for any convolution array. We let $1/G(x)$ be the generating function for a sequence $\{a_n\}_{n=0}^{\infty}$. Then $[1/G(x)]^{k+1}$ is the generating function for the kth convolution of the sequence $\{a_n\}$ and thus the generating function for the kth column of the convolution array for $\{a_n\}$, where the leftmost column is the 0th column.

Theorem 1.2: Let

$$1/G(x) = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function for the sequence $\{a_n\}$. Then the sum of the elements appearing on the p,q-diagonals of the convolution array of $\{a_n\}$ has generating function given by

$$\frac{[G(x)]^{q-1}}{[G(x)]^q - x^p}.$$

Proof: We write the convolution array for $\{a_n\}$ to include the powers of x generated:

$$\begin{array}{ccccc} a_0 & b_0 & c_0 & d_0 & \cdots \\ a_1x & b_1x & c_1x & d_1x & \cdots \\ a_2x^2 & b_2x^2 & c_2x^2 & d_2x^2 & \cdots \\ a_3x^3 & b_3x^3 & c_3x^3 & d_3x^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{array}$$

We call the top-most row the 0th row and the left-most column the 0th column. In order to sum the elements appearing on the p,q-diagonal, we begin at the element a_nx^n, $n = 0, 1, 2, \ldots,$ and move p units up and q units right. We must multiply every qth column, then, successively by x , x^{2p}, x^{3p}, ..., so that the elements summed are coefficients of the same power of x. The generating functions of every qth column, then, when summed, will have the successive sums of elements found along the p,q-diagonals as coefficients of successive powers of x, so that the sum of the adjusted column generators becomes the generating function we seek. But, we notice that we have a geometric progression, so that

$$\frac{1}{G(x)} + \frac{x^p}{[G(x)]^{q+1}} + \frac{x^{2p}}{[G(x)]^{2q+1}} + \cdots = \frac{1/G(x)}{1 - x^p/[1/G(x)]^q} = \frac{[G(x)]^{q-1}}{[G(x)]^q - x^p}.$$

The sums of elements appearing on the p,q-diagonals of Pascal's triangle and generalized Pascal triangles can be found in Hoggatt & Bicknell [2], [3], and Harris & Styles [4].

2. FIBONACCI AND JACOBSTHAL CONVOLUTION ARRAYS

Returning to Pascal's triangle (1.3), since the Jacobsthal polynomials defined in (1.1) have the property that $J_n(x) = 1$ for $x = 0$ and $n = 1, 2, 3,$

..., Pascal's triangle could be considered the convolution triangle for the sequence of numbers $J_n(0)$. Recall that the 2,1-diagonal contains the coefficients of $J_n(x)$ as well as having sum $F_n = J_n(1)$. We now write the convolution array for the sequence of numbers $J_n(1)$, which, of course, is also the Fibonacci convolution array:

	1	1	1	1	1	...
	1	2	3	4	5	...
	2	5	9	14	20	...
(2.1)	3	10	22	40	65	...
	5	20	51	105	190	...
	8	38	111	256	511	...
	...	...	...	...	...	...

Observe that the sums of elements appearing along the 2,1-diagonals are 1, 1, 3, 5, 11, 21, 43, ..., $J_n(2)$, ...

If one now writes the convolution triangle for the numbers $J_n(2)$,

	1	1	1	1	1	...
	1	2	3	4	5	...
	3	7	12	18	25	...
(2.2)	5	16	34	60	95	...
	11	41	99	195	340	...
	21	94	261	...	...	...
	...	...	...	...	...	...

one finds that the sums of elements appearing on the 2,1-diagonals are 1, 1, 4, 7, 19, 40, ..., $J_n(3)$, ...

Finally, we summarize our results below.

Theorem 2.1: When the convolution array for the sequence $J_n(k)$ obtained by letting $x = k$, $k = 0, 1, 2. 3, \ldots$, in the Jacobsthal polynomials $J_n(x)$, $n = 1, 2, 3, \ldots$, is written in rectangular form, the sums of the elements appearing along successive 2,1-diagonals are the numbers $J_n(k + 1)$, and the 2,1-diagonal contains the coefficients of the polynomials $J_n^*(x)$, $n = 1, 2, 3, \ldots$,

$$J_{n+2}^*(x) = J_{n+1}^*(x) + (k + x)J_n^*(x), \quad J_1^*(x) = 1, \quad J_2^*(x) = 1.$$

Proof: The Jacobsthal polynomials are generated by

$$\frac{1}{G(x)} = \frac{1}{1 - x - yx^2} = \sum_{n=0}^{\infty} J_{n+1}(y)x^n$$

From Theorem 1.2, the sums of elements on the 2,1-diagonals have generating function

$$\frac{1}{G(x) - x^2} = \frac{1}{(1 - x - kx^2) - x^2} = \frac{1}{1 - x - (k + 1)x^2},$$

the generating function for the numbers $J_n(k + 1)$.

If one now returns to the array given in (2.2), notice that we also have the convolution array for the Fibonacci numbers, or for the numbers $F_n(1)$. In Pascal's triangle, the 1,1-diagonal contains the coefficients of the Fibonacci polynomials, but in the Fibonacci convolution array, the 1,1-diagonals contain the coefficients of $F_n(x + 1)$, where $F_n(x)$ are the Fibonacci polynomials. If one replaces x by $(x + 1)$ in the display of Fibonacci polynomials given in the introduction, one obtains:

$$1,\ x+1,\ x^2+2x+2,\ x^3+3x^2+5x+3,\ x^4+4x^3+9x^2+10x+5,\ \ldots .$$

If we replace x by $(x + 2)$ in successive polynomials $F_n(x)$, we obtain:

$$1,\ x+2,\ x^2+4x+5,\ x^3+6x^2+14x+12,\ x^4+8x^3+27x^2+44x+29,\ \ldots,$$

where the constant terms are Pell numbers. We next write the convolution array for the Pell numbers, or the numbers $F_n(2)$,

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & \ldots \\ 2 & 4 & 6 & 8 & 10 & \ldots \\ 5 & 14 & 27 & 44 & 65 & \ldots \\ 12 & 44 & 104 & 200 & 340 & \ldots \\ 29 & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & & & & & \end{array} \tag{2.3}$$

and observe that the 1,1-diagonals contain exactly those coefficients of successive polynomials $F_n(x + 2)$. Also, the sums of elements appearing in the 1,1-diagonals are 1, 3, 10, 33, 109, ..., $F_n(3)$, ..., while in the Fibonacci convolution array those sums were given by 1, 2, 5, 12, 29, ..., $F_n(2)$, ..., and in Pascal's triangle those sums were the Fibonacci numbers themselves.

We summarize as follows.

Theorem 2.2: When the convolution array for the sequence $F_n(k)$ obtained by letting $x = k$, $k = 1, 2, 3, \ldots$, in the Fibonacci polynomials $F_n(x)$, $n = 1, 2, 3, \ldots$, is written in rectangular form, the sums of the elements appearing along successive 1,1-diagonals are the numbers $F_n(k+1)$, and the 1,1-diagonals contain the coefficients of the polynomials $F_n(x+k)$.

Proof: The Fibonacci polynomials are generated by

$$\frac{1}{G(x)} = \frac{1}{1 - yx - x^2} = \sum_{n=0}^{\infty} F_{n+1}(y)x^n$$

From Theorem 1.2, the sums of elements on the 1,1-diagonals have generating function

$$\frac{1}{G(x) - x} = \frac{1}{1 - kx - x^2 - x} = \frac{1}{1 - (k + 1)x - x^2},$$

the generating function for the numbers $F_n(k + 1)$.

Rather than using the definition of convolution sequence, one can write all of these arrays by using a simple additive process. For example, each element in Pascal's rectangular array is the sum of the element in the same row, preceding column, and the element above it in the same column. In the Fibonacci convolution array, each element is the sum of the element in the same row, preceding column, and the two elements above it in the same column.

In the convolution array for $\{F_n(k)\}$, the rule of formation is to add the element in the same row, preceding solumn, to k times the element above, and the second element above, as

$$\{F_n(x)\}\ \text{convolution:}\qquad \begin{array}{|c|c|} \hline & w \\ \hline & y \\ \hline x & z \\ \hline \end{array} \qquad z = x + ky + w,\ k = 1, 2, \ldots .$$

The convolution array for $\{J_n(k)\}$ is formed by adding the element in the same row, preceding column, to the element above, and to k times the second element above the desired element, as

$$\{J_n(x)\} \text{ convolution: } \qquad \begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & w \\ \cline{2-2} \multicolumn{1}{c|}{} & y \\ \hline x & z \\ \hline \end{array} \qquad z = x + y + kw,\ k = 0, 1, 2, \ldots.$$

Both additive rules follow immediately from the generating function of the array. For example, for the $\{J_n(k)\}$ convolution, if $G_n(x)$ is the generating function of the nth column, then

$$G_{n+1}(x) = G_1(x)G_n(x) = [1/(1 - x - kx^2)]G_n(x),$$
$$G_{n+1}(x) = G_n(x) + xG_{n+1}(x) + kx^2G_{n+1}(x).$$

As a final example, we proceed to the Tribonacci circumstances. The Tribonacci numbers 1, 1, 2, 4, 7, 13, ..., T_n, ..., given by

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \quad T_0 = 0, \quad T_1 = T_2 = 1, \tag{2.4}$$

appear as the sums of successive 1,1-diagonals of the trinomial triangle written in left-justified form. The trinomial triangle contains as its rows the coefficients of $(1 + x + x^2)^n$, $n = 0, 1, 2, \ldots,$

$$\begin{array}{cccccccccc} 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 3 & 6 & 7 & 6 & 3 & 1 \\ 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \tag{2.5}$$

and the coefficients of the Tribonacci polynomials $T_n(x)$ (see [5], [6]),

$$T_{n+3}(x) = x^2T_{n+2}(x) + xT_{n+1}(x) + T_n(x),$$
$$T_{-1}(x) = T_0(x) = 0, \quad T_1(x) = 1$$

along its 1,1-diagonals. We note that $T_n(1) = T_n$.

If we write instead three other polynomial sequences—$t_n(x)$, $t_n^*(x)$, and $t_n^{**}(x)$—which have the property that $t_n(1) = t_n^*(1) = t_n^{**}(1) = T_n$, we find a remarkable relationship to the convolution array for the Tribonacci numbers.

	$t_{n+3} = xt_{n+2} + t_{n+1} + t_n$	$t_{n+3}^* = t_{n+2}^* + xt_{n+1}^* + t_n^*$
$n = 1$	1	1
$n = 2$	x	1
$n = 3$	$x^2 + 1$	$x + 1$
$n = 4$	$x^3 + 2x + 1$	$2x + 2$
$n = 5$	$x^4 + 3x^2 + 2x + 1$	$x^2 + 3x + 3$
...	...	...

and

$$t^{**}_{n+3} = t^{**}_{n+2} + t^{**}_{n+1} + xt^{**}_{n}$$

$n = 1$	1
$n = 2$	1
$n = 3$	2
$n = 4$	$x + 3$
$n = 5$	$2x + 5$
...	...

We write the convolution array for the Tribonacci numbers:

	1	1	1	1	1	...
	1	2	3	4	5	...
(2.6)	2	5	9	14	20	...
	4	12	25	44	70	...
	7	26	63	...	...	...
	13	56	...			
	24	...				
	...					

If we replace x with $(x + 1)$ in $t_n(x)$, we get 1, $x + 1$, $x^2 + 2x + 2$, $x^3 + 3x^2 + 5x + 4$, ..., whose coefficients appear along the 1,1-diagonals. Putting $(x + 1)$ in place of x in $t^*_n(x)$ gives 1, 1, $x + 2$, $2x + 4$, $x^2 + 5x + 7$, ..., which coefficients are on the 2,1-diagonal, while replacing x by $(x + 1)$ in $t^{**}_n(x)$ makes 1, 1, 2, $x + 4$, $2x + 7$, $5x + 13$, $x^2 + 12x + 24$, ..., which coefficients appear on the 3,1-diagonal. The coefficients of $t^*_n(x + k)$ appear along the 1,1-diagonals of the convolution array for $t^*_n(k)$, and similarly for $t^*_n(x + k)$ and the array for $t^*_n(k)$, and for $t^{**}_n(x + k)$ and $t^{**}_n(k)$.

The Tribonacci convolution array can be generated either by the definition of convolution or by dividing out its generating functions $[1/(1 - x - x^2 - x^3)]^n$ or by the following simple additive process: each element in the array is the sum of the element in the same row but one column left and the three elements above it in the same column, or, schematically,

$$\begin{array}{|c|c|} \hline & w \\ \hline & x \\ \hline & y \\ \hline p & z \\ \hline \end{array} \qquad z = p + w + x + y.$$

Generalizations to generalized Pascal triangles are straightforward.

3. ARRAYS OF NUMERATOR POLYNOMIALS DERIVED FROM FIBONACCI AND JACOBSTHAL CONVOLUTION ARRAYS

In this section, we calculate the generating functions for the rows of the Fibonacci and Jacobsthal convolution arrays of §2. We note that, in each case, the first row is a row of constants; the second row contains elements with a constant first difference; ...; and the ith row forms an arithmetic progression of order $(i - 1)$, $i = 1, 2, \ldots$, with generating function $N_i(x)/(1 - x)^i$. We shall make use of a theorem from a thesis by Kramer [8].

Theorem 57 (Kramer): If generating function

$$A(x) = N(x)/(1 - x)^{r+1}$$

where $N(x)$ is a polynomial of maximum degree r, then $A(x)$ generates an arithmetic progression of order r, and the constant of the progression is $N(1)$.

We calculate the first few row generators for the Fibonacci convolution array (2.1) as

$$\frac{1}{1-x}, \frac{1}{(1-x)^2}, \frac{2-x}{(1-x)^3}, \frac{3-2x}{(1-x)^4}, \frac{5-5x+x^2}{(1-x)^5}, \frac{8-10x+3x^2}{(1-x)^6}.$$

We display the coefficients of the successive numerator polynomials:

	1			
	1			
	2	-1		
	3	-2		
(3.1)	5	-5	1	
	8	-10	3	
	13	-20	9	-1
	21	-38	22	-4
	...	...	...	...

The rising diagonal sums are 1, 1, 2, 2, 3, 3, 4, 4, ..., but if we use absolute values, they become the Tribonacci numbers 1, 1, 2, 4, 7, 13, 24, 44, ... The row sums are all 1, which means, by Theorem 57, that $N_n(1) = 1$, or that the constant of the arithmetic progression of order $(n-1)$ found in the nth row of the Fibonacci convolution array is 1. However, the row sums, using absolute values, are 1, 1, 3, 5, 11, 21, 43, 85, ..., $J_n(2)$, ... Notice that successive columns are formed from successive columns of the Fibonacci convolution array (2.1) itself. We defer proof to the general case.

If one now turns to the convolution array (2.2) for $\{J_n(2)\}$, the first few row generators are

$$\frac{1}{1-x}, \frac{1}{(1-x)^2}, \frac{3-2x}{(1-x)^3}, \frac{5-4x}{(1-x)^4}, \frac{11-14x+4x^2}{(1-x)^5}, \ldots.$$

Displaying the coefficients of the numerator polynomials,

	1				
	1				
	3	-2			
(3.2)	5	-4			
	11	-14	4		
	21	-32	12		
	43	-82	48	-8	
	...	...	...	...	...

we find that the rising diagonal sums are 1, 1, 3, 3, 7, 7, 15, 15, ..., while, taking absolute values, they are 1, 1, 3, 7, 15, 35, 79, ..., where the kth term is formed from the sum of the preceding term and twice the sum of the two terms preceding that, a generalized Tribonacci sequence. Each row sum is again 1. However, using absolute value, the row sums become 1, 1, 5, 9, 29, 65, 181, ..., $J_n(4)$, ... Notice that successive columns are multiples of successive columns of (2.2), the second column being twice the second column of (2.2), the third column four times the original third column, and the fourth column eight times the original fourth column.

Notice that the Fibonacci numbers are also the numbers $J_n(1)$. We state and prove a theorem for the general Jacobsthal case.

Theorem 3.1: Let $J_n^{j+1}(k)$ denote the nth element of the jth convolution of $\{J_n(k)\}$. Let $N_m(x)/(1-x)^m$ denote the generating function of the mth row, $m = 1, 2, \ldots,$ in the convolution array for $\{J_n(k)\}$. Then

$$N_m(x) = \sum_{i=0}^{[(m-1)/2]} (-1)^i k^i J_{m-2i}^{i+1}(k)x^i.$$

Proof: [Note that $J_n^1(k) = J_n(k)$.] From the rule of formation of the convolution array for $\{J_n(k)\}$ derived in §2, the row generators $D_n(x)$ obey

$$D_n(x) = xD_n(x) + D_{n-1}(x) + kD_{n-2}(x) = \frac{1}{1-x}[D_{n-1}(x) + kD_{n-2}(x)] \tag{3.3}$$

$$\frac{N_n(x)}{(1-x)^n} = \frac{1}{1-x}\left[\frac{N_{n-1}(x)}{(1-x)^{n-1}} + \frac{kN_{n-2}(x)}{(1-x)^{n-2}}\right]$$

$$N_n(x) = N_{n-1}(x) + (1-x)kN_{n-2}(x) = N_{n-1}(x) + kN_{n-2}(x) - kxN_{n-2}(x). \tag{3.4}$$

Comparing (3.4) to the original recurrence for $J_n(k)$ and noting that $N_1(x) = N_2(x) = J_1(k) = J_2(k) = 1$, the constant term is given by $N_n(0) = J_n(k)$. The rule of formation of the convolution array can also be stated as

$$J_n^{i+1}(x) = J_{n-1}^{i+1}(k) + kJ_{n-2}^{i+1}(k) + J_n^i(k). \tag{3.5}$$

Let u_n be the coefficient of x in $N_n(x)$. Then

$$u_n = u_{n-1} + ku_{n-2} - kJ_{n-2}(k).$$

If $u_j = -kJ_{j-2}^2(k)$, $j = 3, 4, \ldots, n-1$, then

$$\begin{aligned} u_n &= -kJ_{n-3}^2(k) - k^2J_{n-4}^2(k) - kJ_{n-2}(k) \\ &= -k\big(J_{n-3}^2(k) + kJ_{n-4}^2(k) + J_{n-2}(k)\big) = -kJ_{n-2}^2(k) \end{aligned}$$

by (3.5). Thus, the coefficient of x has the desired form for all $n \geq 3$.

Next, let u_n be the coefficient of x^i and v_n the coefficient of x^{i-1} in $N_n(x)$. If $u_j = (-1)^i J_{j-2i}^{i+1}(k)k^i$ and $v_j = (-1)^{i-1}k^{i-1}J_{j-2i}^i(k)$ for $j = 3, 4, \ldots, n-1$, then

$$\begin{aligned} u_n &= u_{n-1} + ku_{n-2} - kv_{n-2} \\ &= (-1)^i k^i J_{n-1-2i}^{i+1}(k) + k(-1)^i k^i J_{n-2-2i}^{i+1}(k) - k(-1)^{i-1}k^{i-1}J_{n-2i}^i(k) \\ &= (-1)^i k^i\big(J_{n-1-2i}^{i+1}(k) + kJ_{n-2-2i}^{i+1}(k) + J_{n-2i}^i(k)\big) \\ &= (-1)^i k^i\big(J_{n-2i}^{i+1}(k)\big) \end{aligned}$$

by again applying (3.5), establishing Theorem 3.1, except for the number of terms summed. By Theorem 57 [8], $i \leq m$, since the degree of $N_m(x)$ is less than or equal to m. But $J_{m-2i}^{i+1}(k) = 0$ for $[(m-1)/2] < i \leq m$.

By Theorem 3.1, the generating function for the ith column of the numerator polynomial coefficient array for the generating functions of the rows of the convolution array of $\{J_n(k)\}$ is now known to be

$$\frac{k^{i-1}x^{2(i-1)}}{(1 - x - kx^2)^i}.$$

Summing the geometric series

$$\frac{1}{1 - x - kx^2} + \frac{kx^2}{(1 - x - kx^2)^2} + \frac{k^2x^4}{(1 - x - kx^2)^3} + \cdots = \frac{1}{1 - x - (2k)x^2}$$

which proves that the rows' sums, using absolute values, are given by $J_n(2k)$. However, summing for the rows as originally given, we use alternating signs in forming the geometric series, and its sum becomes $1/(1-x)$, so that $N_m(1) = 1$. That is, the ith row is an arithmetic progression of order $(i-1)$ with constant 1 in every one of the arrays for $\{J_n(k)\}$, $k = 1, 2, 3, \ldots$.

Turning to the cases of convolution arrays for the sequences $\{F_n(k)\}$, $k = 1, 2, 3, \ldots$, we look at $F_n(2)$ as in array (2.3). The first few row generators are

$$\frac{1}{1 - x}, \frac{2}{(1 - x)^2}, \frac{5 - x}{(1 - x)^3}, \frac{12 - 4x}{(1 - x)^4}, \frac{29 - 14x + x^2}{(1 - x)^5}, \frac{70 - 44x + 6x^2}{(1 - x)^6}.$$

The array of coefficients for the numerator polynomials is

	1			
	2			
	5	-1		
(3.6)	12	-4		
	29	-14	1	
	70	-44	6	
	...	...	...	...

The row sums are 1, 2, 4, 8, 16, 32, ..., 2^n, ..., and the coefficients of successive columns appear in the original array (2.3). We state the situation for the general case.

Theorem 3.2: Let $F_n^{j+1}(k)$ denote the nth element in the jth convolution of the numbers $\{F_n(k)\}$, $k = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$. Let the generating function of the mth row in the convolution array for $\{F_n(k)\}$ be $N_m^*(x)/(1 - x)^m$, $m = 1, 2, \ldots$. Then

$$N_m^*(x) = \sum_{i=0}^{[(m-1)/2]} (-1)^i F_{m-2i}^{i+1}(k)x^i.$$

The proof is analogous to that of Theorem 3.1 and is omitted in the interest of brevity.

Theorem 3.2 tells us that the ith column of the numerator coefficient array form the generating functions of the rows of the convolution arrays for $F_n(k)$ is given by $(-1)^i x^{2i}/(1 - kx - x^2)^i$. Then, $N_n^*(1)$ is the sum of the rows given by the sum of the geometric series

$$\frac{1}{1 - kx - x^2} - \frac{x^2}{(1 - kx - x^2)^2} + \frac{x^4}{(1 - kx - x^2)^3} - \cdots = \frac{1}{1 - kx},$$

so that $N_n^*(1) = k^{n-1}$. By Theorem 57 [8], the $(i-1)$st order arithmetic progression formed in the ith row of the convolution array for $\{F_n(k)\}$ has constant k^{n-1}, in every one of the arrays, $k = 1, 2, 3, \ldots$.

4. ARRAYS OF SUCCESSIVE JACOBSTHAL AND FIBONACCI POLYNOMIAL SEQUENCES

In [7], Whitford considers an array whose rows are given by successive sequences derived from the Jacobsthal polynomials, such as

(4.1)

k	The sequence $\{J_n(k)\}$										
0	1	1	1	1	1	1	1	1	1	1	...
1	1	1	2	3	5	8	13	21	34	55	...
2	1	1	3	5	11	21	43	85	171	341	...
3	1	1	4	7	19	40	97	217	508	1159	...
4	1	1	5	9	29	65	181	441	1165	2929	...
...					...						

The successive elements in each column are given by 1, 1, $k+1$, $2k+1$, k^2+3k+1, ..., by the recursion relation for $\{J_n(k)\}$. The vertical sequences above are given by

$$J_n(k) = \sum_{r=0}^{n-1}\binom{n-1-r}{r}k^r = \frac{1}{2^{n-1}}\sum_{\substack{r=1\\ r\text{ odd}}}^{n}\binom{n}{r}(4k+1)^{(r-1)/2} \tag{4.2}$$

where n is fixed, $n \geq 1$, and $k = 0, 1, 2, 3, \ldots$ (see [7]).

We now wish to obtain the generating functions for the columns of the array (4.1). Notice that the first two columns are constants, the next two columns have a constant second difference, the next two have a constant third difference, etc. This means that if $D_n(x)$ is the generating function for the nth column, $n = 1, 2, 3, \ldots$, then the denominators of $D_{2m-1}(x)$ and $D_{2m}(x)$ are each given by $(1-x)^m$. We shall again make use of Theorem 57 [8], which was quoted in §3.

One has

$$D_{2m-1}(x) = \frac{r_{2m-1}(x)}{(1-x)^m}, \quad D_{2m}(x) = \frac{r_{2m}(x)}{(1-x)^m},$$

by virtue of

$$\frac{1}{1-x-kx^2} = \sum_{n=0}^{\infty}J_{n+1}(k)x^n.$$

Now, if $J_{n+1}(k)$ has fixed $(n+1)$ and k varies, we generate the columns. If we fix k and let n vary, we generate the rows. $J_{n+1}(k)$ is a polynomial in k with coefficients lying along the 2,1-diagonal of Pascal's triangle. To get the ordinary generating function, we can note that

$$\frac{A_k(x)}{(1-x)^{k+1}} = \sum_{n=0}^{\infty}n^k x^n,$$

where the $A_k(x)$ are the Eulerian polynomials. (See Riordan [9] and Carlitz [10]). Thus, we note that the polynomials $J_{2m-1}(k)$ and $J_{2m}(k)$ are both of the same degree, and we will expect the generating functions to reflect this fact.

From careful scrutiny of the array generation, we see

$$(4.3)\qquad D_{n+2}(x) = D_{n+1}(x) + xD_n'(x).$$

One then breaks this down into two cases:

$$(4.4)\qquad \begin{aligned} D_{2m+2}(x) &= \frac{r_{2m+2}(x)}{(1-x)^{m+1}} = \frac{r_{2m+1}(x)}{(1-x)^{m+1}} + x\frac{d}{dx}\left(\frac{r_{2m}(x)}{(1-x)^m}\right) \\ D_{2m+3}(x) &= \frac{r_{2m+3}(x)}{(1-x)^{m+2}} = \frac{r_{2m+2}(x)}{(1-x)^{m+1}} + x\frac{d}{dx}\left(\frac{r_{2m+1}(x)}{(1-x)^{m+1}}\right) \end{aligned}$$

This leads to two simple recurrences:

$$(4.5)\qquad \begin{aligned} r_{2m+2}(x) &= r_{2m+1}(x) + x(1-x)r_{2m}'(x) + mxr_{2m}(x) \\ r_{2m+3}(x) &= (1-x)r_{2m+2}(x) + x(m+1)r_{2m+1}(x) + x(1-x)r_{2m+1}'(x) \end{aligned}$$

The first fifteen polynomials $r_n(x)$ are:

n	$r_n(x)$	row sums $r_n(1)$
1	1	0!
2	1	1!
3	1	1!
4	$1 + x$	2!
5	$1 + 2x - x^2$	2!
6	$1 + 5x$	3!
7	$1 + 9x - 3x^2 - x^3$	3!
8	$1 + 17x + 7x^2 - x^3$	4!
9	$1 + 29x + 11x^2 - 17x^3$	4!
10	$1 + 50x + 76x^2 - 6x^3 - x^4$	5!
11	$1 + 83x + 164x^2 - 100x^3 - 29x^4 + x^5$	5!
12	$1 + 138x + 516x^2 + 110x^3 - 45x^4$	6!
13	$1 + 226x + 1121x^2 - 50x^3 - 571x^4 - 8x^5 + x^6$	6!
14	$1 + 370x + 2843x^2 + 2344x^3 - 421x^4 - 98x^5 + x^6$	7!
15	$1 + 602x + 6071x^2 + 4956x^3 - 5249x^4 - 1430x^5 + 89x^6$	7!

We observe that $r_n(1) = [n/2]!$, where $[x]$ is the greatest integer contained in x. This follows immediately by taking $x = 1$ in (4.5) to make a proof by mathematical induction. By Theorem 57 [8], $r_n(1)$ also is the constant of the arithmetic progression formed by the elements in the nth column of the Jacobsthal polynomial array (4.1). There is a pleasant surprise in the second column of the numerator polynomials $r_n(x)$, whose generating function is

$$1/[(1-x-x^2)(1-x)(1-x^2)].$$

The sequence of coefficients is 0, 0, 0, 1, 2, 5, 9, 17, 29, 50, 83, 138, 226, 370, 602, ..., u_r, ..., $r = 1, 2, 3, \ldots$. We can prove from the recurrence relation that

$$(4.6)\qquad \begin{aligned} u_{2k-1} &= F_{2k-1} - k \\ u_{2k} &= F_{2k} - k \end{aligned}$$

as r is odd or even. By returning to (4.5), we can write a recurrence for the u_r simply by looking for those terms which contain multiples of x only, so that

$$u_{2m+2} = u_{2m+1} + u_{2m} + m$$

$$u_{2m+3} = (u_{2m+2} - 1) + (m + 1) + u_{2m+1} = u_{2m+2} + u_{2m+1} + m$$

Since we know that (4.6) holds for $r = 1, 2, \ldots, 15$, we examine u_{2m+2} and u_{2m+3}, assuming that (4.6) holds for all $r < 2m + 2$. Then

$$u_{2m+2} = \left(F_{2m+1} - (m + 1)\right) + (F_{2m} - m) + m = F_{2m+2} - (m + 1)$$

$$u_{2m+3} = \left(F_{2m+2} - (m + 1)\right) + \left(F_{2m+1} - (m + 1)\right) + m = F_{2m+3} - (m + 2),$$

so that (4.6) holds for all integers r by mathematical induction.

To determine the relationship between elements appearing in the third column of the numerator polynomial array, examine (4.5) to write only those terms which contain multiples of x^2. Letting the coefficient of x^2 in $r_n(x)$ be v_n, we obtain

$$v_{2m+2} = v_{2m+1} + 2v_{2m} + (m - 1)u_{2m}$$

$$v_{2m+3} = v_{2m+2} + 2v_{2m+1} + mu_{2m+1} - u_{2m+2}$$

which, when combined with (4.6), gives us

$$\begin{aligned} v_{2m+2} &= v_{2m+1} + 2v_{2m} - u_{2m} + mF_{2m} - m^2 \\ &= v_{2m+1} + 2v_{2m} - (m - 1)(F_{2m} - m) \end{aligned} \tag{4.7}$$

$$\begin{aligned} v_{2m+3} &= v_{2m+2} + 2v_{2m+1} - u_{2m+2} + mF_{2m+1} - 2t_m \\ &= v_{2m+2} + 2v_{2m+1} + (m + 1)F_{2m+1} - F_{2m+3} - (m + 1)^2 \end{aligned}$$

where $t_m = m(m + 1)/2$, the mth triangular number.

Continuing to the fourth column, if the coefficient of x^3 in $r_n(x)$ is w_n, we can write

$$w_{2m+2} = w_{2m+1} + 3w_{2m} + (m - 2)v_{2m}$$

$$w_{2m+3} = w_{2m+2} + 3w_{2m+1} + (m - 1)v_{2m+1} - v_{2m+2},$$

and so on.

Now, if we wish to generate the columns of the numerator polynomials array, it is easy enough to write the generators for the second column if we take two cases. To write the generating function for 1, 5, 17, 50, 138, ..., u_{2n}, ..., since this is the sequence of second partial sums of the alternate Fibonacci numbers 1, 3, 8, 21, 55, ..., the generating function is

$$1/[(1 - 3x - x^2)(1 - x)^2,$$

except to use it properly, we must replace x by x^2, so that

$$\frac{1}{(1 - 3x^2 + x^4)(1 - x^2)^2} = \sum_{n=0}^{\infty} u_{2n+4}x^{2n}$$

Now, the generating function for u_{2k-1} results from combining the known generators for F_{2k-1} and for the positive integers.

Since

$$\frac{1 - x}{1 - 3x + x^2} = 1 + 2x + 5x^2 + 13x^3 + 34x^4 + \cdots$$

and

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots,$$

$$\frac{1-x}{1-3x+x^2} - \frac{1}{(1-x)^2} = \frac{2x^2 - x^3}{(1-3x+x^2)(1-x)^2} = \sum_{n=1}^{\infty}(F_{2n-1} - n)x^{n-1}.$$

To adjust the powers of x, first replace x by x^2 and then multiply each side by x, obtaining finally

$$\frac{2x^5 - x^7}{(1-3x^2+x^4)(1-x^2)^2} = \sum_{n=1}^{\infty}(F_{2n-1} - n)x^{2n-1} = \sum_{n=1}^{\infty}u_{2n-1}x^{2n-1}.$$

On the other hand, if one writes the array whose rows are given by successive sequences derived from the Fibonacci polynomials,

(4.8)

k	The sequence $\{F_n(k)\}$								
1	1	1	2	3	5	8	13	21	...
2	1	2	5	12	29	70	169	308	...
3	1	3	10	33	109	360	1189	3927	...
4	1	4	17	72	305	1292	5273	22384	...
...					...				

The successive elements in each column are given by 1, k, k^2+1, k^3+2k, k^4+3k^3+1, ..., by the recursion relation for $F_n(k)$, $k = 1, 2, 3, \ldots$. The vertical sequences above are given by

$$(4.9)\qquad F_n(k) = \sum_{r=0}^{n-1}\binom{n-1-r}{r}k^{n-2r-1}$$

where n is fixed, $n \geq 1$, and $k = 1, 2, 3, \ldots$, or by

$$\frac{1}{1-kx-x^2} = \sum_{n=0}^{\infty}F_{n+1}(k)x^n,$$

which generates the rows for k fixed, $n = 1, 2, 3, \ldots$, and the columns for n fixed, $k = 1, 2, 3, \ldots$.

As before, we wish to generate the columns. We observe, since the nth column has a constant $(n-1)$st difference, that the denominators of the column generators will be $(1-x)^n$, $n = 1, 2, 3, \ldots$.

If we let $D_n^*(x)$ be the generating function for the nth column, and let

$$(4.10)\qquad D_n^*(x) = \frac{r_n^*(x)}{(1-x)^n},$$

this time we find that

$$D_{n+2}^*(x) = xD_{n+1}^{*\prime}(x) + D_n^*(x);$$

$$(4.11)\qquad r_{n+2}^*(x) = x(n+1)r_{n+1}^*(x) + x(1-x)r_{n+1}^{*\prime}(x) + (1-x)^2r_n^*(x).$$

We list the first few numerator polynomials $r_n^*(x)$:

n	$r_n^*(x)$	$r_n^*(1)$
1	1	$0!$
2	x	$1!$
3	$1 - x + 2x^2$	$2!$
4	$3x + 3x^3$	$3!$
5	$1 + 14x^2 + 4x^3 + 5x^4$	$4!$
6	$8x + 22x^2 + 60x^3 + 22x^4 + 8x^5$	$5!$
7	$1 + 6x + 99x^2 + 244x^3 + 279x^4 + 78x^5 + 13x^6$	$6!$
8	$21x + 240x^2 + 1251x^3 + 2016x^4 + 1251x^5 + 240x^6 + 21x^7$	$7!$
9	$1 + 25x + 715x^2 + 5245x^3 + 14209x^4 + 14083x^5 + 5329x^6 + 679x^7 + 34x^8$	$8!$
...	...	...

We find that $r_n^*(1) = (n - 1)!$, and that the coefficient of the highest power of x in $r_n^*(x)$ is F_n. It would also appear that the coefficients of x are alternate Fibonacci numbers in even-numbered rows. In fact, D. Garlick [11] observed that, if u_n is the coefficient of the linear term in $r_n^*(x)$, then

$$\begin{aligned} u_{2k} &= F_{2k} \\ u_{2k-1} &= F_{2k-1} - (2k - 1), \end{aligned}$$

which can be proved from the recurrence relation by induction.

Let c_n be the constant term in $r_n^*(x)$. By studying (4.11) carefully to find first, constant terms only, and then just the linear terms, we can write

$$\begin{aligned} c_{n+2} &= c_n \\ u_{n+2} &= (n + 1)c_{n+1} + u_{n+1} + u_n - 2c_n. \end{aligned} \tag{4.13}$$

Since $c_1 = 1$ and $c_2 = 0$, $c_{2k+1} = 1$ and $c_{2k} = 0$. Assume that (4.12) is true for all $n \leq 2k$. Then, taking $n = 2k - 1$ in (4.13),

$$\begin{aligned} u_{2k+1} &= (2k)c_{2k} + u_{2k} + u_{2k-1} - 2c_{2k-1} \\ &= 0 + F_{2k} + F_{2k-1} - (2k - 1) - 2 = F_{2k+1} - (2k + 1). \end{aligned}$$

Similarly, from (4.13) for $n = 2k$,

$$\begin{aligned} u_{2k+2} &= (2k + 1)c_{2k+1} + u_{2k+1} + u_{2k} - 2c_{2k} \\ &= (2k + 1) + F_{2k+1} - (2k + 1) + F_{2k} - 0 = F_{2k+2}, \end{aligned}$$

so that (4.12) holds for all integers $k > 0$.

Continuing, let v_n be the coefficient of x^2 in $r_n^*(x)$. By looking only at coefficients of x^2 in (4.11), we have

$$\begin{aligned} v_{n+2} &= (n + 1)u_{n+1} + 2v_{n+1} - u_{n+1} + v_n - 2u_n + c_n \\ &= 2v_{n+1} + v_n + nu_{n+1} - 2u_n + c_n, \end{aligned}$$

which, combined with (4.12), makes

$$v_{2k+2} = 2v_{2k+1} + v_{2k} + 2k\big(F_{2k+1} - (2k + 1)\big) - 2F_{2k}$$

$$v_{2k+1} = 2v_{2k} + v_{2k-1} + (2k - 1)F_{2k} - 2\big(F_{2k-1} - (2k - 1)\big) + 1.$$

Now, to prove that the coefficient of the highest power of x is F_n, we let the coefficient of the highest power of x in $r_n^*(x)$ be h_n. As before, (4.11) gives us

$$h_{n+2} = (n + 1)h_{n+1} - nh_{n+1} + h_n = h_{n+1} + h_n.$$

Since $h_1 = 1$ and $h_2 = 1$, $h_n = F_n$.

Further, it was conjectured by Hoggatt and proved by Carlitz [12] that $r_{2n}^*(x)$ is a symmetric polynomial. Note that this also gives the linear term of $r_{2n}^*(x)$ the value F_{2n} since we have just proved that the highest power of x has F_n for a coefficient.

5. INFINITE SEQUENCES OF DETERMINANT VALUES

In [13] and [14], sequences of $m \times m$ determinants whose values are binomial coefficients were found when Pascal's triangle was imbedded in a matrix. Here, we write infinite sequences of determinant values of $m \times m$ determinants found within the rectangular arrays displayed throughout this paper. We will apply

Eves' Theorem: Consider a determinant of order n whose ith row (column) ($i = 1, 2, \ldots, n$) is composed of any n successive terms of an arithmetic progression of order ($i - 1$) with constant a_i. Then the value of the determinant is the product $a_1 a_2 \ldots a_n$.

Consider the convolution array for the powers of 2 as given in (1.5). Each row is an arithmetic progression of order ($i - 1$) and with constant 2^{i-1}, $i = 1, 2, 3, \ldots$. Thus, the determinant of any square $m \times m$ array taken to include elements from the first row of (1.5) is $2^0 2^1 2^2 \ldots 2^{m-1} = 2^{m(m-1)/2}$. Further, noticing that each element in the array is 2^{i-1} times the element of Pascal's triangle in the corresponding position in the ith row, $i = 1, 2, \ldots$, we can apply the theorems known about Pascal's triangle from [13] and [14]. However, if we form the convolution triangle for powers of k, then each element in the ith row is k^{i-1} times the corresponding element in the ith row of Pascal's triangle written in rectangular form, $i = 1, 2, \ldots$. Thus, applying the known theorems for Pascal's triangle, we could immediately evaluate determinants correspondingly placed in the powers of k convolution triangle.

Also, we notice that the convolution array for the sequence $\{J_n(k)\}_{n=0}^{\infty}$, $k = 0, 1, 2, 3, \ldots$, has its rows in arithmetic progressions of order $(i-1)$ with constant 1, $i = 1, 2, \ldots$, while the convolution array for the sequence $\{F_n(k)\}_{n=0}^{\infty}$, $k = 1, 2, 3, \ldots$, has its rows in arithmetic progressions of order $(i-1)$ with constant k^{i-1}, $i = 1, 2, \ldots$. From these remarks, we have the theorem given below.

Theorem 5.1: Form the $m \times m$ matrix A such that it contains m consecutive rows of the original array, with its first row the first row of the original array, and m consecutive columns of the original array with its first column the jth column of the original array. In the convolution array for the sequence $\{J_n(k)\}_{n=0}^{\infty}$, $k = 0, 1, 2, \ldots$, $\det A = 1$. In the convolution array for the sequence $\{F_n(k)\}_{n=0}^{\infty}$, $k = 1, 2, 3, \ldots$, or in the convolution array for the powers of k, $\det A = k^{m(m-1)/2}$.

Determinants whose values are binomial coefficients also appear within these arrays. To apply the results of [13] and [14], we must first express our convolution arrays in terms of products of infinite matrices. Let the rectangular convolution array for $\{F_n(k)\}$ be imbedded in an infinite matrix $\mathcal{F}_k$, and similarly, let $\mathcal{J}_k$ be the infinite matrix formed from the convolution array for $\{J_n(k)\}$. Let P be the infinite matrix formed by Pascal's triangle written in rectangular form. Consider the convolution array for the powers of k, written in rectangular form. Each successive 1,1-diagonal contains the coefficients of $(x + k)^n$. Form the matrix A_k such that the coefficients of $(k + n)^n$ appear in its columns on and beneath the main diagonal, and the matrix

B_k in exactly the same way, except use the coefficients of $(1 + kx)^n$. Then, $A_kP = \mathcal{F}_k$ and $B_kP = \mathcal{J}_k$. We illustrate, using 5 x 5 matrices, for $k = 2$:

$$A_2P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 4 & 0 & 0 & \cdots \\ 0 & 0 & 4 & 8 & 0 & \cdots \\ 0 & 0 & 1 & 12 & 16 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 4 & 6 & 8 & 10 & \cdots \\ 5 & 14 & 27 & 44 & 65 & \cdots \\ 12 & 44 & 104 & 200 & 340 & \cdots \\ 29 & 121 & 366 & 810 & 1555 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \mathcal{F}_2$$

$$B_2P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 4 & 1 & 0 & \cdots \\ 0 & 0 & 4 & 6 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 3 & 7 & 12 & 18 & 25 & \cdots \\ 5 & 16 & 34 & 60 & 95 & \cdots \\ 11 & 41 & 99 & 195 & 340 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \mathcal{J}_2$$

Using the methods of [13] and [14], since the generating function for the jth column of A_k is $[x(k+x)]^{j-1}$ while the jth column of P is $1/(1-x)^j$, the jth column of A_kP is $1/[1-x(k+x)]^j = [1-kx-x^2]^j$, where we recognize the generating functions for the columns of the convolution array for $\{F_n(k)\}$, so that $A_kP = \mathcal{F}_k$. Similarly, since the jth column of B_k is generated by $[x(1 + kx)]^{j-1}$, B_kP is generated by $1/[1-x(1+kx)]^{j-1} = 1/[1 - x - kx]^{j-1}$, so that $B_kP = \mathcal{J}_k$.

Each submatrix of $\mathcal{J}_k$ taken with its first row anywhere along the first row or second row of $\mathcal{J}_k$ is the product of a similarly placed submatrix of P and a matrix with unit determinant. The case for $\mathcal{F}_k$ is similar, except that an $m \times m$ submatrix of P is multiplied by an $m \times m$ matrix whose determinant is $k^{m(m-1)/2}$. Since we know how to evaluate determinants of submatrices of P [13], [14], we write

Theorem 5.2: Form an $m \times m$ matrix B from m consecutive rows and columns of the original array by starting its first row along the second row of the original array and its first column along the jth column of the original array. In the convolution array for the sequence $\{J_n(k)\}_{n=0}^{\infty}$, $k = 0, 1, 2, \ldots$, $\det B = \binom{j-1+m}{m}$. In the convolution array for the sequence $\{F_n(k)\}_{n=0}^{\infty}$, $k = 1, 2, 3, \ldots$, or in the convolution array for the powers of k, $\det B = k^{m(m-1)/2}\binom{j-1+m}{m}$.

We could extend the results of Theorem 5.1 to apply to any convolution array for a sequence with first term 1 and second term k, since Hoggatt and Bergum [15] have shown that such convolution arrays always have the ith row an arithmetic progression of order $(i-1)$ with constant k. It is conjectured that Theorem 5.2 also holds for the convolution array of any increasing sequence whose first term is 1 and second term is k.

Proceeding to the array formed from the Jacobsthal sequences themselves, as given in (4.1), the nth column is an arithmetic progression of order $[(n-1)/2]$, where $[x]$ is the greatest integer contained in x. That makes determinants of value zero very easy to find. Any determinant formed with its first column the first, second, or third column of the original array containing any m consecutive rows of m consecutive columns, $m > 3$, is zero. Det A = det B = 0 whenever $m > j$, for matrices A and B formed as in Theorems 5.1 and 5.2. However, determinants formed from m consecutive rows taken from alternate columns have value $(0!)(1!)(2!) \ldots (m-1)!$ or $(1!)(2!) \ldots (m!)$ depending upon whether one takes the first column and then successive odd columns or begins with the second column and then successive even columns.

Similarly, the array (4.8) formed of the sequences $\{F_n(k)\}_{n=0}^{\infty}$, $k = 1, 2, 3, \ldots$, has its ith column an arithmetic progression of order $(i-1)$ with constant $(i-1)!$, so that any determinant formed from any m consecutive rows of the first m columns has determinant $(0!)(1!) \ldots (m-1)!$.

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