ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months of publication of the problems.

H-302 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX*

Let c be a constant and define the sequence $\langle a_n \rangle$ by $a_0 = 1$, $a_1 = 2$, and $a_n = 2a_{n-1} + ca_{n-2}$ for $n \geq 2$. Determine the sequence $\langle b_n \rangle$ for which

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k.$$

H-303 *Proposed by Paul Bruckman, Concord, CA*

If $0 < s < 1$, and n is any positive integer, let

$$(1) \quad H_n(s) = \sum_{k=1}^{n} k^{-s},$$

and

$$(2) \quad \theta_n(s) = \frac{n^{1-s}}{1-s} - H_n(s).$$

Prove that $\lim\limits_{n\to\infty} \theta_n(s)$ exists, and find this limit.

H-304 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

(a) Show that there is a unique partition of the positive integers, N, into two sets, A_1 and A_2, such that

$$A_1 \cup A_2 = n, \qquad A_1 \cap A_2 = \emptyset,$$

and no two distinct elements from the same set add up to a Lucas number.

(b) Show that every positive integer, M, which is not a Lucas number is the sum of two distinct elements of the same set.

H-305 *Proposed by Martin Schechter, Swarthmore College, Swarthmore, PA*

For fixed positive integers, m, n, define a Fibonacci-like sequence as follows:

$$S_1 = 1, \; S_2 = m, \; S_k = \begin{cases} mS_{k-1} + S_{k-2} & \text{if } k \text{ is even} \\ nS_{k-1} + S_{k-2} & \text{if } k \text{ is odd.} \end{cases}$$

(Note that for $m = n = 1$, one obtains the Fibonacci numbers.)

(a) Show the Fibonacci-like property holds that if j divides k then S_j divides S_k and in fact that $(S_q, S_r) = S_{(q,r)}$ where (,) = g.c.d.

(b) Show that the sequence obtained when

$$[m = 1, n = 4] \text{ and when } [m = 1, n = 8],$$

respectively, have only the element 1 in common.

H-306 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

(a) Prove that the system S,

$$a + b = F_p, \quad b + c = F_q, \quad c + a = F_r,$$

cannot be solved in positive integers if F_p, F_q, F_r are positive Fibonacci numbers.

(b) Likewise, show that the system T,

$$a + b = F_p,\ b + c = F_q,\ c + d = F_r,\ d + e = F_s,\ e + a = F_t,$$

has no solution under the same conditions.

(c) Show that if F_p is replaced by any positive non-Fibonacci integer, then S and T have solutions.

If possible, find necessary and sufficient conditions for the system U,

$$a + b = F_p,\ b + c = F_q,\ c + d = F_r,\ d + a = F_s,$$

to be solvable in positive integers.

SOLUTIONS

Indifferent

H-276 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Show that the sequence of Bell numbers, $\{B_i\}_{i=0}^{\infty}$ is invariant under repeated differencing.

$$B_0 = 1,\ B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, \quad (n \geq 0).$$

Solution by Paul S. Bruckman, Concord CA

The following exponential inversion formula is well known:

$$(1) \quad f(n) = \sum_{k=0}^{n} \binom{n}{k} g(k) \quad \text{iff} \quad g(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k).$$

Setting $g(n) = B_n$ and $f(n) = B_{n+1}$ in (1), we obtain the result:

$$(2) \quad B_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} B_{k+1}, \quad n = 0, 1, 2, \ldots .$$

However, the right member of (2) is precisely $\Delta^n B_1$. Hence,

(3) $\Delta^n B_1 = B_n, \quad n = 0, 1, 2, \ldots .$

Also solved by N. Johnson and the proposer.

OLDER STUMPERS!

H-254 *Proposed by R. Whitney, Lock Haven State College, Lock Haven, PA*

Consider the Fibonacci-Pascal-type triangle given below.

$$\begin{array}{ccccccccc}
 & & & & F_1 & & & & \\
 & & & F_1 & & F_1 & & & \\
 & & F_1 & & F_2 & & F_1 & & \\
 & F_1 & & F_3 & & F_3 & & F_1 & \\
F_1 & & F_4 & & F_6 & & F_4 & & F_1 \\
- & - & - & - & - & - & - & - & -
\end{array}$$

Find a formula for the row sums of this array.

H-260 *Proposed by H. Edgar, San Jose State University, San Jose, CA*

Are there infinitely many subscripts, n, for which F_n or L_n are prime?

H-271 *Proposed by R. Whitney, Lock Haven State College, Lock Haven, PA*

Define the binary dual, D, as follows:

$$D = \left\{ t \,\middle|\, t = \prod_{i=0}^{n} (a_i + 2i); \quad a_i \in \{0, 1\}; \quad n \geq 0 \right\}.$$

Let $\overline{D}$ denote the complement of D with respect to the set of positive integers. Form a sequence, $\left\{S_n\right\}_{n=1}^{\infty}$, by arranging $\overline{D}$ in increasing order. Find a formula for S_n.

(Note: The elements of D result from interchanging + and × in a binary number. D contains items like $2^n \cdot n!$, $3 \cdot 2^{n-1} n!$, ...)

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