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A CONJECTURE IN GAME THEORY

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We consider a team composed of n players, with each member playing the same r games, $G_1, G_2, \ldots, G_r$. We assume that each game G_j has two possible outcomes, success and failure, and that the probability of success in game G_j is equal to p_j for each player. We let X_{ij} be equal to one (1) if player i has a success in game j and let X_{ij} be equal to zero (0) if player i has a failure in game j. We assume throughout this paper that the random variables X_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, r$ are independent.

Let S_{jn} denote the total number of successes in the jth game. We define the point-value of a team to be

$$\Psi_n = \min_{1 \le j \le r} S_{jn}.$$

This means that the point-value of a team is equal to the minimum number of successes in any particular game. Clearly,

$$P\{S_{jn} = m\} = \binom{n}{m} p_j^m (1 - p_j)^{n-m}, \quad m = 0, 1, 2, \ldots, n,$$

and

$$\begin{aligned} (1) \qquad E[\Psi_n] &= \sum_{k=0}^{n} kP\{\Psi_n = k\} = \sum_{k=0}^{n-1} P\{\Psi_n > k\} \\ &= \sum_{k=0}^{n-1} P\{S_{1n} > k, S_{2n} > k, \ldots, S_{rn} > k\} \\ &= \sum_{k=0}^{n-1} \prod_{j=1}^{r} P\{S_{jn} > k\} \\ &= \sum_{k=0}^{n-1} \prod_{j=1}^{r} \sum_{m=k+1}^{n} \binom{n}{m} p_j^m (1 - p_j)^{n-m}. \end{aligned}$$

It follows from the definition of Ψ_n that the expected point-value for a team is an increasing function of n, i.e.,

$$E[\Psi_n] \le E[\Psi_{n+1}], \quad n = 1, 2, 3, \ldots .$$

Since a team can add players in order to increase its expected point-value, it seems reasonable to define the score to be the expected point-value per player. Namely, we denote the score by

$$W_n = \frac{1}{n} E[\Psi_n].$$

Thus, from (1), we obtain

$$W_n = \frac{1}{n}\sum_{k=0}^{n-1} \prod_{j=1}^{r} \sum_{m=k+1}^{n} \binom{n}{m} p_j^m (1 - p_j)^{n-m}. \tag{2}$$

It is not obvious from (2) how the score varies as the number of players increases. We now prove that W_n is a strictly increasing function of n in the special case $r = 2$ and $p_1 = p_2$. We first prove three lemmas, which are also of independent interest.

Lemma 1: Let a team be composed of j players, with each member playing the same two games, G_1 and G_2. Let the probability of success for each player in both games G_1 and G_2 be equal and be denoted by p. Let $u_j = P\{S_{1j} = S_{2j}\}$, for all positive integers j. Then

$$\frac{1}{2\Pi}\int_0^{2\Pi} |p + qe^{i\theta}|^{2j} d\theta = u_j,$$

where $q = 1 - p$.

Proof: Using the fact that

$$P\{S_{ij} = m\} = \binom{j}{m} p^m (1 - p)^{j-m}, \ m = 0, 1, 2, \ldots, j, \ i = 1, 2,$$

and the independence of the random variables S_{1j} and S_{2j}, we obtain

$$u_j = \sum_{m=0}^{j} \left[\binom{j}{m}\right]^2 p^{2m} (1 - p)^{2(j-m)}, \ j = 1, 2, 3, \ldots . \tag{3}$$

We note that if f is the polynomial $f(z) = \sum_{m=0}^{j} a_m z^m$, then

$$\frac{1}{2\Pi}\int_0^{2\Pi} |f(e^{i\theta})|^2 d\theta = \sum_{m=0}^{j} a_m^2. \tag{4}$$

We now apply the binomial expansion and (4) to the function $f(z) = (p + qz)^j$, where j is a positive integer. The binomial expansion yields

$$f(z) = (p + qz)^j = \sum_{m=0}^{j} \left[\binom{j}{m} p^m q^{j-m}\right] z^{j-m},$$

and using (3) and (4), we obtain

$$\frac{1}{2\Pi}\int_0^{2\Pi} |p + qe^{i\theta}|^{2j} d\theta = \sum_{m=0}^{j} \left[\binom{j}{m}\right]^2 p^{2m} q^{2(j-m)} = u_j. \tag{5}$$

Lemma 2: Let $r = 2$, $p_1 = p_2$, and $u_j = P\{S_{1j} = S_{2j}\}$, for all positive integers j. Then $u_j < u_{j-1}$.

Proof: Since

$$|p + qe^{i\theta}|^2 \leq 1, \text{ for } 0 \leq \theta \leq 2\Pi$$

and

$$|p + qe^{i\theta}|^2 < 1, \text{ for } 0 < \theta < 2\Pi,$$

the desired result follows from (5).

Lemma 3: Let $u_j = P\{S_{1j} = S_{2j}\}$, for all positive integers j and let $u_0 = 1$. Let $d_j = \Psi_{j+1} - \Psi_j$, $j = 0, 1, 2, \ldots$, and let $\Psi_0 = 0$. Then

$$E[d_j] = u_j p^2 + (1 - u_j)p. \tag{6}$$

Proof: Clearly, d_j can assume only the values 0 and 1 with the following probabilities:

$$P\{d_j = 0\} = 1 - [u_j p^2 + (1 - u_j)p],$$
$$P\{d_j = 1\} = u_j p^2 + (1 - u_j)p.$$

Since $E[d_j] = 0 \cdot P\{d_j = 0\} + 1 \cdot P\{d_j = 1\}$, we obtain the desired result.

Theorem: Let a team be composed of n players, with each member playing the same two games, G_1 and G_2. Let the probability of success for each player in both games G_1 and G_2 be equal and be denoted by p. Then

$$W_n < W_{n+1},\ n = 1, 2, 3, \ldots .$$

Proof: Using the definition of W_n, we obtain

$$W_{n+1} - W_n = E\left[\frac{\Psi_{n+1}}{n+1} - \frac{\Psi_n}{n}\right] = \frac{1}{n(n+1)} E[n(\Psi_{n+1} - \Psi_n) - \Psi_n]. \tag{7}$$

Using d_j, as defined in Lemma 3, and noting that $\Psi_n = \sum_{j=0}^{n-1} d_j$, (7) reduces to

$$W_{n+1} - W_n = \frac{1}{n(n+1)} E\left[nd_n - \sum_{j=0}^{n-1} d_j\right].$$

Using (6), we obtain

$$W_{n+1} - W_n = \frac{1}{n(n+1)}\left[n(u_n p^2 + (1 - u_n)p) - \sum_{j=0}^{n-1} (u_j p^2 + (1 - u_j)p)\right].$$

Thus, to prove that $W_n < W_{n+1}$, it suffices to show that

$$n(u_n p^2 + (1 - u_n)p) - \sum_{j=0}^{n-1} (u_j p^2 + (1 - u_j)p) > 0. \tag{8}$$

Proving inequality (8) is equivalent to showing that

$$nu_n - \sum_{j=0}^{n-1} u_j = \sum_{j=1}^{n} j(u_j - u_{j-1}) < 0. \tag{9}$$

Since (9) follows from Lemma 2, we conclude that

$$W_n < W_{n+1},\ n = 1, 2, 3, \ldots .$$

It is the author's conjecture that in the general case discussed in the beginning of this paper ($r > 2$ and p_1 not necessarily equal to p_2) that W_n is a strictly increasing function of n, too. The above proven theorem and some elementary numerical computations suggest the truth of this statement, but the author has not been able to supply a complete proof.

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