A GENERALIZATION OF WYTHOFF'S GAME

V. E. HOGGATT, JR., MARJORIE BICKNELL-JOHNSON, and RICHARD SARSFIELD
San Jose State University, San Jose, CA 95192

Wythoff's game is a variation of Nim, a two-pile game in which each player removes counters in turn until the winner takes the last counter. The safe-pairs generated in the solution of Wythoff's game have many properties interesting in themselves, and are related to the canonical Zeckendorf representation of an integer using Fibonacci numbers. In Nim, the strategy is related to expressing the numbers in each pile in binary notation, or representing them by powers of 2. Here, the generalized game provides number sequences related to the canonical Zeckendorf representation of integers using Lucas numbers.

1. INTRODUCTION: WYTHOFF'S GAME

Wythoff's game is a two-pile game where each player in turn follows the rules:

(1) At least one counter must be taken;
(2) Any number of counters may be removed from one pile;
(3) An equal number of counters may be removed from each pile;
(4) The winner takes the last counter.

The strategy is to control the number of counters in the two piles to have a safe position, or one in which the other player cannot win. Wythoff devised a set of "out of a hat" safe positions

$$(1,2),\ (3,5),\ (4,7),\ (6,10),\ \ldots,\ (a_n,b_n).$$

It was reported by W. W. Rouse Ball [1] that

$$a_n = [n\alpha] \quad \text{and} \quad b_n = [n\alpha^2] = a_n + n,$$

where α is the Golden Section Ratio, $\alpha = (1+\sqrt{5})/2$, and $[n]$ is the greatest integer not exceeding n.

More recently, Nim games have been studied by Whinihan [2] and Schwenk [3], who showed that the safe positions were found from the unique Zeckendorf representation of an integer using Fibonacci numbers, but did not consider properties of the number pairs themselves. Properties of Wythoff pairs have been discussed by Horadam [4], Silber [5], [6], and Hoggatt and Hillman [7].

For completeness, we will list the first forty Wythoff pairs and some of their properties that we will generalize. Also, we will denote the Fibonacci numbers by F_n, where $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, and the Lucas numbers by L_n, $L_n = F_{n-1} + F_{n+1}$. (See Table 1.)

Generation of Wythoff pairs:

I. Begin with $a_1 = 1$. Always take $b_n = a_n + n$, and take a_k as the smallest integer not yet appearing in the table.

II. Let $B = \{b_n\}$ and $A = \{a_n\}$. Then A and B are disjoint sets whose union is the set of positive integers, and A and B are self-generating. B is generated by taking $b_1 = 2$ and $b_{n+1} = b_n + 2$ if $n \in B$ or $b_{n+1} = b_n + 3$ if $n \notin B$. A is generated by taking $a_1 = 1$ and $a_{n+1} = a_n + 2$ if $n \in A$ or $a_{n+1} = a_n + 1$ if $n \notin A$.

TABLE 1

THE FIRST FORTY WYTHOFF PAIRS

n	a_n	b_n	n	a_n	b_n	n	a_n	b_n	n	a_n	b_n
1	1	2	11	17	28	21	33	54	31	50	81
2	3	5	12	19	31	22	35	57	32	51	83
3	4	7	13	21	34	23	37	60	33	53	86
4	6	10	14	22	36	24	38	62	34	55	89
5	8	13	15	24	39	25	40	65	35	56	91
6	9	15	16	25	41	26	42	68	36	58	94
7	11	18	17	27	44	27	43	70	37	59	96
8	12	20	18	29	47	28	45	73	38	61	99
9	14	23	19	30	49	29	46	75	39	63	102
10	16	26	20	32	52	30	48	78	40	64	104

Properties of Wythoff pairs:

(1.1) $a_k + k = b_k$

(1.2) $a_n + b_n = a_{b_n}$

(1.3) $a_{a_n} + 1 = b_n$

(1.4) $a_n = 1 + \delta_3 F_3 + \delta_4 F_4 + \cdots + \delta_k F_k$, where $\delta_i \in \{0, 1\}$

(1.5) $b_n = 2 + \delta_4 F_4 + \delta_5 F_5 + \cdots + \delta_m F_m$, where $\delta_i \in \{0, 1\}$

(1.6) $a_{a_n+1} - a_{a_n} = 2$ and $a_{b_n+1} - a_{b_n} = 1$

(1.7) $b_{a_n+1} - b_{a_n} = 3$ and $b_{b_n+1} - b_{b_n} = 2$

(1.8) $a_n = [n\alpha]$ and $b_n = [n\alpha^2]$

2. GENERALIZED WYTHOFF NUMBERS

First, we construct a table of numbers which are generalizations of the safe Wythoff pair numbers (a_n, b_n) of Section 1. We let $A_n = 1$, and take

$$B_n = A_n + d_n, \text{ where } d_n \neq B_k + 1$$

(that is, $d_{n+1} = d_n + 1$ when $d_n \neq B_k$ or $d_{n+1} = d_n + 2$ when $d_n = B_k$, and $d_1 = 2$). Notice that before, $b_n = a_n + n$; here, we are removing any integer that is expressible by $B_k + 1$. We let $C_n = B_n - 1$. To find successive values of A_n, we take A_n to be the smallest integer not yet used for A_i, B_i, or C_i in the table. We shall find many applications of these numbers, and also show that they are self-generating. In Table 2, we list the first twenty values.

We next derive some properties of the numbers A_n, B_n, and C_n. First, A_n, B_n, and C_n can all be expressed in terms of the numbers a_n and b_n of the Wythoff pairs from Section 1. Note that A_{2k} is even, and A_{2k+1} is odd, an obvious corollary of Theorem 2.1.

Theorem 2.1:

(i) $A_n = 2a_n - n$;

(ii) $B_n = a_n + 2n = b_n + n = a_{a_n} + 1 + n$;

(iii) $C_n = a_n + 2n - 1 = b_n + n - 1 = a_{a_n} + n$.

TABLE 2

THE FIRST TWENTY GENERALIZED WYTHOFF NUMBERS

n	A_n	B_n	d_n	C_n	n	A_n	B_n	d_n	C_n
1	1	3	2	2	11	23	39	16	38
2	4	7	3	6	12	26	43	17	42
3	5	10	5	9	13	29	47	18	46
4	8	14	6	13	14	30	50	20	49
5	11	18	7	17	15	33	54	21	53
6	12	21	9	20	16	34	57	23	56
7	15	25	10	24	17	37	61	24	60
8	16	28	12	27	18	40	65	25	64
9	19	32	13	31	19	41	68	27	67
10	22	36	14	35	20	44	72	28	71

Proof of Theorem 2.1: First, we prove (ii) and (iii). Consider the set of integers $\{1, 2, 3, \ldots, B_n\}$, which contains n Bs and n Cs, since $C_n = B_n - 1$, and j As, where $A_j = B_n - 2$. Since A, B, and C are disjoint sets, B_n is the sum of the number of Bs, the number of Cs, and the number of As, or,

$$B_n = n + n + j = 2n + j,$$

$$C_n = 2n - 1 + j.$$

Note that

$$A_{a_n} = C_n - 1 = B_n - 2, \text{ for } n = 1, 2, 3, 4, 5.$$

Assume that $A_{a_n} = C_n - 1$, or that the number of As less than B_n is $j = a_n$.

$$A_{a_n} = C_n - 1$$

$$A_{a_n} + 1 = C_n \neq A_{a_n + 1}$$

$$A_{a_n} + 2 = C_n + 1 = B_n \neq A_{a_n + 1}$$

$$A_{a_n} + 3 = C_n + 2$$

but $A_{a_n} + 3 = A_{a_n + 1}$, since the As differ by 1 or 3 and they do not differ by 1; and $C_n + 2 = C_{n+1} - 1$ or $C_n + 2 = C_{n+1} - 2$ since the Cs differ by 3 or 4. If $A_{a_n + 1} = C_{n+1} - 1$ and $a_n + 1 = a_{n+1}$, we are through; this occurs only when $n = b_k$ by (1.6). If $n \neq b_k$, then $n = a_k$. Note that the As differ by 1 or 3. Since $A_{a_n} + 3 = A_{a_n + 1}$, $A_{b_k} + 3 \neq A_{b_k + 1}$, since $a_n \neq b_k$ for any k. Thus,

$$A_{b_k} + 1 = A_{b_k + 1}.$$

Now, if $A_{a_n + 1} = C_{n+1} - 2$, then

$$A_{a_n + 1} + 1 = C_{n+1} - 1.$$

But, $n = a_k$, so $a_{a_k} + 1 = b_k = a_n + 1$, so

$$A_{a_n + 1} + 1 = A_{b_k} + 1 = A_{b_k + 1} = A_{a_n + 2} = A_{a_{n+1}}$$

by (1.6), so that again $A_{a_{n+1}} = C_{n+1} - 1$. Thus, by the axiom of mathematical induction, $j = a_n$, and we have established (ii) and (iii).

Now, we prove (i). Either $A_{k+1} = A_k + 1$ or $A_{k+1} = A_k + 3$. From Table 2, observe that $A_1 = 2a_1 - 1$ and $A_2 = 2a_2 - 2$. Assume that $A_k = 2a_k - k$. When $k = b_j$, we have

$$\begin{aligned} A_{k+1} &= A_k + 1 \\ &= (2a_k - k) + 1 \\ &= 2(a_k + 1) - (k + 1) \\ &= 2a_{k+1} - (k + 1) \end{aligned}$$

by (1.6). If $k \neq b_j$, then $k = a_j$, and we have

$$\begin{aligned} A_{k+1} &= A_k + 3 \\ &= (2a_k - k) + 3 \\ &= 2(a_k + 2) - (k + 1) \\ &= 2a_{k+1} - (k + 1) \end{aligned}$$

again by (1.6), establishing (i) by mathematical induction.

Following immediately from Theorem 2.1, and from its proof, we have

Theorem 2.2:

(i) $A_{b_n+1} - A_{b_n} = 1$ and $A_{a_n+1} - A_{a_n} = 3$

(ii) $B_{b_n+1} - B_{b_n} = 3$ and $B_{a_n+1} - B_{a_n} = 4$

(iii) $C_{b_n+1} - C_{b_n} = 3$ and $C_{a_n+1} - C_{a_n} = 4$

(iv) $A_{a_n} = a_n + 2n - 2$

(v) $A_{a_n} = C_n - 1$

Theorem 2.3:

$$B_n = [n\alpha\sqrt{5}],\ C_n = [n\alpha\sqrt{5}] - 1,\ \text{and } A_n = 2[n\alpha] - n,$$
where $\alpha = (1 + \sqrt{5})/2$.

Proof: By Theorem 2.1 and property (1.8),

$$\begin{aligned} B_n &= a_n + 2n = [n\alpha] + 2n = [n(\alpha + 2)] \\ &= [n(5 + \sqrt{5})/2] = [n\alpha\sqrt{5}]. \end{aligned}$$

$$A_n = 2a_n - n = 2[n\alpha] - n.$$

Theorem 2.4:

(i) $B_m + B_n \neq A_j$

(ii) $C_m + C_n \neq C_j$

(iii) $B_m + C_n \neq B_j$

(iv) $A_m + B_n \neq C_j$

Proof: (ii) was proved by A. P. Hillman [8] as follows. Let $C_r = a_r + 2r - 1$ and $C_s = a_s + 2s - 1$. Then

$$C_r + C_s = a_r + a_s + 2(r + s) - 2 = C_{r+s} + (a_r + a_s - a_{r+s} - 1),$$

but $a_r = [r\alpha]$ and $a_s = [s\alpha]$ and $a_{r+s} = [(r+s)\alpha]$. However

$$[x] + [y] - [x + y] = 0 \text{ or } -1,$$

so that

$$a_r + a_s - a_{r+s} - 1 = -1 \text{ or } -2,$$

making

$$C_r + C_s = C_{r+s} - 1 \quad \text{or} \quad C_r + C_s = C_{r+s} - 2,$$

but members of the sequence $\{C_k\}$ have differences of 3 or 4 only, so

$$C_{r+s} - 1 \neq C_k, \text{ and } C_{r+s} - 2 \neq C_k, \text{ for any } k.$$

The proof of (i) is similar:

$$\begin{aligned} B_m + B_n &= a_m + 2m + a_n + 2n \\ &= a_m + a_n + 2(m + n) \\ &= a_m + a_n + (b_{m+n} - a_{m+n}) + (m + n) \\ &= (a_m + a_n - a_{m+n}) + \big(b_{m+n} + (m + n)\big) \\ &= (0 \text{ or } -1) + B_{m+n}. \end{aligned}$$

Thus,

$$B_m + B_n = B_{m+n} \quad \text{or} \quad B_m + B_n = B_{m+n} - 1 = C_{m+n},$$

and

$$B_m + B_n \neq A_j, \text{ for any } j.$$

Now, to prove (iii),

$$\begin{aligned} B_m + C_n &= (a_m + 2m) + (a_n + 2n - 1) \\ &= a_m + a_n + (m + n) + (m + n) - 1 \\ &= a_m + a_n + (b_{m+n} - a_{m+n}) + (m + n) - 1 \\ &= (a_m + a_n - a_{m+n}) + \big(b_{m+n} + (m + n) - 1\big) \\ &= (0 \text{ or } -1) + C_{m+n}. \end{aligned}$$

Thus,

$$B_m + C_n = C_{m+n} \quad \text{or} \quad B_m + C_n = C_{m+n} - 1 = B_{m+n} - 2,$$

but consecutive members of B_j differ by 3 or by 4, so

$$B_{m+n} - 2 \neq B_j, \text{ and } B_m + C_n \neq B_j.$$

Lastly, to prove (iv), either

$$C_{j+1} - C_j = 3 \quad \text{or} \quad C_{j+1} - C_j = 4,$$

and

$$A_m \neq C_j,\ A_m \neq B_j, \text{ for any } j.$$

If $A_m \neq C_j$ and $A_m \neq C_j + 1 = B_j$, then either

$$A_m = C_j - 1 \quad \text{or} \quad A_m = C_j + 2 = B_j + 1.$$

If $A_m = C_j - 1$, then

$$A_m + B_n = C_j - 1 + B_n,$$

which equals

$$C_{j+n} - 1 \neq C_k \quad \text{or} \quad C_{j+n} - 2 \neq C_k, \text{ by the proof of (iii).}$$

If $A_m = B_j + 1$, then

$$A_m + B_n = B_j + 1 + B_n$$

which equals either

$$B_{j+n} + 1 = C_{j+n} - 2 \neq C_k \quad \text{or} \quad C_{j+n} + 1 \neq C_k,$$

by the proof of (i).

The proof of Theorem 2.4 gives us immediately three further statements relating A_n, B_n, and C_n, the first three parts of Theorem 2.5.

Theorem 2.5:

(i) $B_m + B_n = B_{m+n}$ or $B_m + B_n = C_{m+n}$

(ii) $C_m + C_n = C_{m+n} - 1$ or $C_m + C_n = C_{m+n} - 2$

(iii) $B_m + C_n = C_{m+n}$ or $B_m + C_n = C_{m+n} - 1$

(iv) $A_m + A_n = A_{m+n}$ or $A_m + A_n = A_{m+n} - 2$

Proof of (iv):

$$\begin{aligned} A_m + A_n &= 2a_m - m + 2a_n - n \\ &= 2b_m - 3m + 2b_n - 3n \\ &= 2b_m + 2b_n - 3(m + n) \\ &= 2b_m + 2b_n - 3(b_{m+n} - a_{m+n}) \\ &= (2b_m + 2b_n - 2b_{m+n}) - b_{m+n} + 3a_{m+n} \\ &= (2b_m + 2b_n - 2b_{m+n}) - (a_{m+n} + (m + n) + 3a_{m+n}) \\ &= (2b_m + 2b_n - 2b_{m+n}) + 2a_{m+n} - (m + n) \\ &= (0 \text{ or } -2) + A_{m+n}, \end{aligned}$$

so that

$$A_m + A_n = A_{m+n} \quad \text{or} \quad A_m + A_n = A_{m+n} - 2.$$

Finally, we can write some relationships between A_n, B_n, and C_n when the subscripts are the same.

Theorem 2.6:

(i) $A_n + B_n = A_{b_n}$

(ii) $A_n + C_n = B_{a_n}$

(iii) $A_n + B_n + C_n = C_{b_n}$

(iv) $A_{b_n} + C_n = C_{b_n}$

(v) $B_{a_n} + B_n = C_{b_n}$

Proof: Proof of (i):

$$\begin{aligned} A_n + B_n &= 2a_n - n + a_n + 2n \\ &= 2a_n + b_n \\ &= a_n + a_{b_n} \\ &= 2a_{b_n} + (a_n - a_{b_n}) = 2a_{b_n} - b_n = A_{b_n}. \end{aligned}$$

Proof of (ii): Using (1.1), (1.3), (1.2), and Theorem 2.1,

$$\begin{aligned} A_n + C_n &= 2a_n - n + a_n + 2n - 1 \\ &= 2b_n + a_n - n - 1 \end{aligned}$$

$$= b_n + a_n - n + a_{a_n}$$
$$= 2a_n + a_{a_n} = B_{a_n}$$

Proof of (iii):

$$\begin{aligned}A_n + B_n + C_n &= 2a_n - n + a_n + 2n + a_n + 2n - 1\\ &= a_n + 3(a_n + n) - 1\\ &= a_n + 3b_n - 1\\ &= (a_n + b_n) + 2b_n - 1\\ &= a_{b_n} + 2b_n - 1\\ &= C_{b_n}\end{aligned}$$

by (1.2) and Theorem 2.1.

Note that (iv) and (v) are just combinations of (i) with (iii), and (ii) with (iii).

Notice that there are eighteen possible ways to add two of the As, Bs, or Cs to obtain an A_k, B_j, or C_i. $A_m + A_n = A_{m+n}$ or $A_m + A_n = A_{m+n} - 2$, so that $A_m + A_n = A_k$ or $A_m + A_n = B_j$ or $A_m + A_n = C_i$ for suitable k, j, and i. $A_m + B_n = A_k$ or $A_m + B_n = B_j$ for suitable k and j, but $A_m + B_n \neq C_i$ for any i. $A_m + C_n = A_k$ or $A_m + C_n = B_j$ or $A_m + C_n = C_i$ for suitable values of k, j, and i as readily found in Table 2. Since $B_m + B_n = B_{m+n}$ or $B_m + B_n = C_{m+n}$, solutions exist for $B_m + B_n = B_j$ and $B_m + B_n = C_i$, but $B_m + B_n \neq A_k$ for any k. Since $B_m + C_n = C_{m+n}$ or $B_m + C_n = C_{m+n} - 1$, solutions exist for $B_m + C_n = C_i$ and for $B_m + C_n = A_k$, but $B_m + C_n \neq B_j$ for any j. Lastly, $C_m + C_n = C_{m+n} - 1$ or $C_m + C_n = C_{m+n} - 2$, so it is possible to solve $C_m + C_n = A_k$ and $C_m + C_n = B_j$, but $C_m + C_n \neq C_i$ for any i.

3. LUCAS REPRESENTATIONS OF THE NUMBERS A_n, B_n, AND C_n

The numbers A_n, B_n, and C_n can be represented uniquely as sums of Lucas numbers L_n, where $L_0 = 2$, $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$. Since the Lucas numbers 2, 1, 3, 4, 7, 11, ..., are complete, one could show that $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ cover the positive integers and are disjoint. See [9] and [10]. We write $A = \{A_n\} = \{1, 4, 5, 8, 11, \ldots\}$, numbers in the form

$$A_n = 1 + \delta_2 L_2 + \delta_3 L_3 + \cdots + \delta_m L_m,\ \delta_i \in \{0, 1\}$$

in their natural order; and $B = \{B_n\} = \{3, 7, 10, 14, 18, \ldots\}$,

$$B_n = 3 + \delta_3 L_3 + \delta_4 L_4 + \cdots + \delta_m L_m,\ \delta_i \in \{0, 1\}$$

in their natural order, and $C = \{C_n\} = \{2, 6, 9, 13, 17, \ldots\}$, which are numbers of the form

$$C_n = 2 + \delta_3 L_3 + \delta_4 L_4 + \cdots + \delta_m L_m,\ \delta_i \in \{0, 1\}.$$

The union of A_n, B_n, and C_n is the set of positive integers, and the sets are disjoint. One notes immediately that $B_n = C_n + 1$ because any choice of δs in the set C can be used in the set B so that, for each C_n, there is an element of B which is one greater. Also, each A_n is one greater than a B_j or one less than a C_j. Also, A_ns may be successive integers. A viable approach is to let all the positive integers representable using 1, 3, 4, 7, ... , in Zeckendorf form be classified as having the lowest nonzero binary digit in the even place 3 Lucas Zeckendorf $\boxed{1\,|\,0}$, while 4 is in an odd $\boxed{1\,|\,0\,|\,0}$. This clearly makes A and B distinct sets, since the Zeckendorf representation is unique. Set C consists of the numbers which must use a 2, making C distinct from either A or B, since the positive integers have a distinct and unique representation if no two consecutive Lucas numbers from {2, 1, 3, 4, 7, ...} are used and $L_0 = 2$ and $L_2 = 3$ are not to be used together in any representation.

$C = \{2, 6, 9, 13, 17, 20, \ldots\}$, are the positive integers that are not representable by {1, 3, 4, 7, ...}, the Lucas numbers when $L_0 = 2$ is deleted. The sequence $\{B_n\} = \{3, 7, 10, 14, \ldots\}$ occurs in the solution to the International Olympiad 1977, problem 2 [11], which states:

> Given a sequence of real numbers such that the sum of seven consecutive terms is negative, and the sum of eleven consecutive terms is positive, show that the sequence has a finite (less than 17) number of terms. A solution with sixteen terms does exist:
>
> $$5, -5, -13, 5, 5, 5, -13, 5, 5, -13, 5, 5, 5, -13, 5, 5$$
>
> We note that −13 occurs at positions 3, 7, 10, 14, ...

We know that every positive integer has a Zeckendorf representation in terms of 2, 1, 3, 4, 7, 11, ..., and a second canonical representation such that

$$A \xrightarrow{f^*} B\ ,$$

where f^* merely advances the subscripts on the Zeckendorf representation of A (odd position) to a number from B (even position). One needs a result on lexicographical ordering: If, in comparing the Lucas Zeckendorf representation of M and N from the higher-ordered binary digits, the place where they first differ has a one for M and a zero for N, then $M > N$. Clearly, under f^*, the lexicographical ordering is preserved.

Now, look at the positive integers, and below them write the number obtained by shifting the Lucas subscripts by one upward:

$$\begin{array}{rcccccccccccccc}
n & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
f^*(n) & = & 3 & 1 & 4 & 7 & 10 & 8 & 11 & 14 & 12 & 15 & 18 & 21 & 19
\end{array}$$

$$\Delta f^*(n) = \quad -2 \quad 3 \quad 3 \quad 3 \quad -2 \quad 3 \quad 3 \quad -2 \quad 3 \quad 3 \quad 3 \quad -2$$

Note that the −2 occurs between $n = C_k$ and $n = C_k - 1$; all the other differences $\Delta f^*(n) = f^*(n+1) - f^*(n) = 3$. Now, of course, $1 \to 3$ so that normally the difference of the images of two successive integers is 3, but $2 \to 1$ and $1 \to 3$, so that those integers C_k which require a 2 in their representation always lose 2 in the forward movement of the subscripts.

Now from the positive integers we remove $B_n + 1$, as this is an unpermitted difference; these numbers 4, 8, 11, 15, 19, ..., are A_js immediately after a B_k. Those A_ns remaining in the new set are the second A_n of each adjacent pair. Since $A_{b_n+1} - A_{b_n} = 1$, it follows that

$$A_{b_n+1} \neq B_n + 1,$$

but rather

$$A_{b_n+1} = C_j - 1.$$

The numbers $A_{b_n} = B_m + 1$, $A_{a_n+1} = B_j + 1$, and other A_ns which are of the form $B_s + 1$ are gone. Only $A_{b_n+1} = C_j - 1$ are left in the set, which is {5, 12, 16, 23, 30, ...}, and they are in the set $\{B_n - A_n\}$ and also in $\{A_n\}$. We now wish to look at the fact that each $C_n - 2$ has been removed whenever $C_n - 1$ remains in the set. This opens up an interval difference of six in each case. For instance,

$$\begin{array}{cccccccccc} 3 & 4 & 5 & 6 = C_n & \qquad & 10 & 11 & 12 & 13 = C_n \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 7 & 10 & 8 & & 15 & 18 & 21 & 19 \end{array}$$

difference 6 6

Now, without changing anything else, by giving each element of $\{5, 12, 16, 23, 30, \ldots, A_{b_n+1}, \ldots\}$ an image which is five smaller. Each such number uses a one, $1 = L_1$, in the Zeckendorf representation. Replace this L_1 by $-L_{-1}$. Now, regardless of whatever else is present in this Lucas representation, formerly $1 = L_1 \to L_2 = 3$, but now $1 = -L_{-1} \to -2 = -L_0$, so that the difference in images is 5.

$$A_{b_n+1} \xrightarrow{f^*} M$$

$$A_{b_n+1} \xrightarrow{f} M - 5$$

Now, all of the rest of the differences were 3 when the difference in the objects was 1. The differences in the images were -2 only when the objects were C_n and $C_n - 1$. Now, with 1, 4, 8, 11, 15, ..., $B_n + 1$, ..., removed $(B_0 = 0)$, and each image of the object set $\{5, 12, 16, 23, 30, \ldots, A_{b_n+1}, \ldots\}$ replaced by five less, we now find that if the object numbers differ by 1, their images differ by 3, and if the object numbers differ by two, the image numbers differ by one. Thus, if from the object set $M > N$, then the image of M is greater than the image of N under the mapping f of increasing the Lucas number subscripts by one. This shows that the mapping from $\{\Delta_n\} = \{B_n - A_n\}$ into $\{A_j\}$ is such that $\Delta_n \xrightarrow{f} A_n$. Further, under f^*, operating on the Zeckendorf form, $A_n \xrightarrow{f^*} B_n$ because of the lexicographic mapping. Clearly, f is not lexicographic over the positive integers but over the set $\{\Delta_n\}$ where $\{5, 12, 16, 30, \ldots\}$ have been put into special canonical form.

We note that the set $\{\Delta_n\}$ is all numbers of the form

$$\Delta_n = 2 + \delta_1 L_1 + \delta_2 L_2 + \cdots$$

in their natural order, where $\delta_i \in \{0, 1\}$. Since

$$\{C_n\} = \{2, 6, 9, 13, 17, \ldots\}$$

cannot be made using $\{1, 3, 4, 7, 11, 18, \ldots\}$, it follows that

$$\{C_n + 2\} = \{4, 8, 11, 15, \ldots\} = \{B_n + 1\}$$

cannot be so represented. We have thrown these numbers out of the original set of positive integers. What is left is the set so representable. Thus, Δ_n are the numbers of that form in natural order. The number 1 does not appear in $\{\Delta_n\}$. Now, if in $\{\Delta_n\}$ we replace each Lucas number by one with the next higher subscript, then we get all the numbers of the form

$$1 + \delta_1 L_2 + \delta_2 L_3 + \delta_3 L_4 + \cdots$$

in their natural order since we have carefully made the construction so numbers of the image set are out of their natural order. Thus,

$$\Delta_n \xrightarrow{f} A_n \quad \text{and} \quad A_n \xrightarrow{f^*} B_n.$$

4. WYTHOFF'S LUCAS GAME

The Lucas generalization of Wythoff's game is a two-pile game for two players with the following rules:

(1) At least one counter must be taken;
(2) Any number of counters may be taken from one pile;
(3) An equal number of counters may be taken from each pile;
(4) One counter may be taken from the smaller pile, and two from the larger pile;
(5) All counters may be taken if the numbers of counters in the two piles differ by one (hence, a win);
(6) The winner takes the last counter.

Let H be the pile on the left, and G on the right, so that (H_n, G_n) are to be safe pairs.

$$H_{A_n} = A_{a_n} \qquad G_{A_n} = B_{a_n}$$

$$H_{B_n} = A_{b_n} \qquad G_{B_n} = B_{b_n}$$

$$H_{C_n} = C_{a_n} \qquad G_{C_n} = C_{b_n}$$

where (a_n, b_n) is a safe pair for Wythoff's game, and A_n, B_n, and C_n are the numbers of Section 2.

Now remember that (H_{A_n}, G_{A_n}) and (H_{B_n}, G_{B_n}) had all the differences except numbers of the form $B_n + 1$.

$$C_{b_n} - C_{a_n} = B_n + 1 = A_{a_n+1}.$$

The only difference not in $\{G_n - H_n\}$ is one; hence, Rule 5. The differences in the Hs are 1, 2, or 3, and the differences in the Gs are 1, 3, or 4. It is not difficult to see that the rules change a safe position into an unsafe position.

Next, the problem is to prove that using the rules, an unsafe pair can be made into a safe pair. Strategy to win the Wythoff-Lucas game follows.

Suppose you are left with (c, d) which is an unsafe pair. Without loss of generality, take $c < d$.

1. If $c = H_k$ and $d > G_k$, then choose s so that $d - s = G_k$. (Rule 2.)
2. If $c = H_k$ and $d < G_k$ and $d - c = \Delta_m < \Delta_k = G_k - C_k$, where Δ_m appears in the list of differences earlier, then choose s so that

$$d - s = c + \Delta_m - s = c - s + \Delta_m = H_m + \Delta_m = G_m. \quad \text{(Rule 3.)}$$

3. If $c = H_k$ and $d < G_k$ and $d - c = \Delta_m < \Delta_k$ but Δ_m does not appear earlier in the list of differences, that is, $H_m > H_k$, then we need some results before we can proceed.

Lemma 1: $C_{b_n} - C_{a_n} = B_n + 1 = G_{C_n} - H_{C_n}$.

Proof: By Theorem 2.1,

$$\begin{aligned} C_{b_n} - C_{a_n} &= (a_{b_n} + 2b_n - 1) - (a_{a_n} + 2a_n - 1) \\ &= (a_n + b_n + 2b_n - 1) - (a_{a_n} + 1 + 2a_n - 2) \\ &= a_n + 3b_n - 1 - b_n - 2a_n + 2 \\ &= a_n + 2(b_n - a_n) + 1 \\ &= a_n + 2n + 1 \\ &= B_n + 1. \end{aligned}$$

Lemma 2: $G_{B_n} - H_{B_n} = B_{b_n} - A_{b_n} = B_n$.

Proof: By Theorem 2.1,

$$\begin{aligned} B_{b_n} - A_{b_n} &= a_{b_n} + 2b_n - (2a_{b_n} - b_n) \\ &= 3b_n - a_{b_n} \\ &= 3b_n - (a_n + b_n) \\ &= 2b_n - a_n \\ &= 2(a_n + n) - a_n \\ &= a_n + 2n \\ &= B_n \end{aligned}$$

The original list of $\Delta_n = G_n - H_n$ was steadily increasing functions of n. However, with the insertion of $G_{C_n} - H_{C_n} = \Delta_{C_n} = B_n + 1$, the next higher

$$G_{C_n+1} - H_{C_n+1} = G_{B_n} - H_{B_n} = B_{b_n} - A_{b_n} = B_n$$

by Lemma 2. Thus, if $\Delta_m < \Delta_k$ while $H_m > H_k$, this cannot be the case, except when $c = H_k = H_{C_n}$ while $d = G_{C_n} - 1$. The proper response is to subtract one from c and subtract two from d to finish case 3; we also need Lemma 3, which follows from Theorems 2.1 and 2.2:

Lemma 3: $c - 1 = H_{C_n} - 1 = C_{a_n} - 1 = A_{b_n - 1}$;

$$d - 2 = G_{C_n} - 3 = C_{b_n} - 3 = B_{b_n - 1}.$$

The pair $(H_{C_n - 1}, G_{C_n - 1}) = (A_{b_n - 1}, B_{b_n - 1})$ is a safe pair which is obtained by using Rule 4.

4. If $c = G_k$, then if $d > H_k$, choose s so that $d - s = H_k$. (Rule 2.)
5. If $c = G_k$ and $d < H_k$, follow the procedures of cases 2 and 3.
6. If $c = d - 1$, then take all the counters by Rule 5.

Since G_k and H_k cover the integers, cases 1-6 give every possible choice of c and d, $c \neq d$. If $c = d$, then take all the counters and hence win, by Rule 3.

Some comment should be made about why each legal play from a safe pair results in an unsafe pair. We begin with the safe pair (H_k, G_k) and apply each rule.

(a) If from (H_k, G_k) we subtract $s > 0$ from either (Rule 2), then since (H_k, G_k) are a related pair, changing either one without the other results in an unsafe pair.

(b) If from (H_k, G_k) we subtract $s > 0$ from each (Rule 3), then the difference $\Delta_k = G_k - H_k$ is preserved, but the difference Δ_k is unique to the safe pair; hence, changing H_k and G_k but keeping the difference Δ_k the same results in an unsafe pair.

(c) To investigate Rule 4, we need some results on the differences of the sequences H_n and G_n separately.

Lemma 4: The differences of the H_n sequence are 1, 2, or 3:

(i) $H_{C_n+1} - H_{C_n} = 2;$

(ii) $H_{B_n+1} - H_{B_n} = 1;$

(iii) $H_{A_{b_n}+1} - H_{A_{b_n}} = 3;$

(iv) $H_{A_{a_n}+1} - H_{A_{a_n}} = 1.$

Proof: We refer to Theorem 2.1 and the results of Section 1 repeatedly.

(i) $H_{C_n+1} = H_{B_n} = A_{b_n}$ and $H_{C_n} = C_{a_n}$, so

$$H_{C_n+1} - H_{C_n} = A_{b_n} - C_{a_n} = (2a_{b_n} - b_n) - (a_{a_n} + 2a_n - 1)$$
$$= (2a + 2b - b_n) - b_n - 2a_n + 2 = 2.$$

(ii) $H_{B_n} = A_{b_n}$, so

$$H_{B_n+1} = H_{A_{a_n+1}} = A_{a_{a_n+1}} = A_{a_{a_n}+2} = A_{b_n+1} = A_{b_n} + 1,$$

and

$$H_{B_n+1} - H_{B_n} = A_{b_n} + 1 - A_{b_n} = 1.$$

(iii) $H_{A_{b_n}} = A_{a_{b_n}}$ and $H_{A_{b_n}+1} = H_{A_{b_n+1}} = A_{a_{b_n+1}} = A_{a_{b_n}+1} = A_{a_{b_n}} + 3$, so

$$H_{A_{b_n}+1} - H_{A_{b_n}} = 3.$$

(iv) $H_{A_{a_n}+1} = H_{C_n} = C_{a_n} = a_{a_n} + 2a_n - 1$

$$= (b_n - 1) + 2a_n - 1 = a_{b_n} + a_n - 2;$$

$$H_{A_{a_n}} = A_{a_{a_n}} = 2a_{a_{a_n}} - a_{a_n} = 2(a_{a_{a_n}+1} - 2) - a_{a_n}$$
$$= 2a_{b_n} - 4 - (a_{a_n} + 1) + 1 = 2a_{b_n} - b_n - 3$$
$$= a_{b_n} + (a_n + b_n) - b_n - 3 = a_{b_n} + a_n - 3.$$

Thus,

$$H_{A_{a_n}+1} - H_{A_{a_n}} = (a_{b_n} + a_n - 2) - (a_{b_n} + a_n - 3) = 1.$$

We now conclude that the G sequence does not have a difference of two; in fact, the differences in the Gs are always 1, 3, or 4.

Lemma 5: $G_{k+1} - G_k \neq 2$ and

(i) $G_{C_n+1} - G_{C_n} = 1;$

(ii) $G_{C_n} - G_{C_n-1} = 3;$

(iii) $G_{C_n+2} - G_{C_n+1} = 3;$

(iv) $G_{A_{b_n}+1} - G_{A_{b_n}} = 4.$

Proof: By construction, the differences $\Delta_n = B_n - A_n$ in natural order cover all the positive integers except numbers of the form $B_n + 1$, but $(H_{C_n}, G_{C_n}) = (C_{a_n}, C_{b_n})$ is such that $\Delta_{C_n} = B_n + 1$.

We now cite some obvious results (see Lemmas 1, 2, and 3).

$$\Delta_{C_n} = B_n + 1,\ \Delta_{C_n+1} = B_n,\ \Delta_{C_n-1} = B_n - 1;$$

therefore,

$$\Delta_{C_n+1} - \Delta_{C_n} = -1,\quad \Delta_{C_n} - \Delta_{C_n-1} = 2, \text{ and } \Delta_{C_n+2} - \Delta_{C_n+1} = 2;$$

but

$$\Delta_{m+1} - \Delta_m = 1, \text{ otherwise.}$$

$$(G_{C_n+1} - H_{C_n+1}) - (G_{C_n} - H_{C_n}) = -1;$$

thus,

(i) $G_{C_n+1} - G_{C_n} = H_{C_n+1} - H_{C_n} - 1 = (2 - 1) = 1.$

Since $H_{C_n} - H_{C_n-1} = 3$, $(G_{C_n} - H_{C_n}) - (G_{C_n-1} - H_{C_n-1}) = 2$, making

(ii) $G_{C_n} - G_{C_n-1} = H_{C_n} - H_{C_n-1} + 2 = 3.$

Next, $H_{B_n+1} - H_{B_n} = H_{C_n+2} - H_{C_n+1} = 1$, and $\Delta_{C_n+2} - \Delta_{C_n+1} = 2$, so that

$$G_{C_n+2} - H_{C_n+2} - (G_{C_n+1} - H_{C_n+1}) = 2,$$

or

$$G_{C_n+2} - G_{C_n+1} - (H_{C_n+2} - H_{C_n+1}) = 2,$$

so that, finally,

(iii) $G_{C_n+2} - G_{C_n+1} = (H_{C_n+2} - H_{C_n+1}) + 2 = 1 + 2 = 3.$

The fourth case has $\Delta_{A_{b_n}+1} - \Delta_{A_{b_n}} = 1$, which means that

$$G_{A_{b_n}+1} - H_{A_{b_n}+1} - (G_{A_{b_n}} - H_{A_{b_n}}) = 1,$$

so that

$$G_{A_{b_n}+1} - G_{A_{b_n}} - (H_{A_{b_n}+1} - H_{A_{b_n}}) = 1,$$

making

(iv) $G_{A_{b_n}+1} - G_{A_{b_n}} = (H_{A_{b_n}+1} - H_{A_{b_n}}) + 1 = 3 + 1 = 4.$

The final conclusion is that no difference of Gs equals two, concluding the proof of Lemma 5.

We now can finish case (c), the investigation of Rule 4. The play of subtracting one from H_k and two from G_k does leave an unsafe pair.

(d) If (H_k, G_k) is a safe pair, then the difference between H_k and G_k is never one, so Rule 5 will not apply.

We have found that applying the rules to a safe pair always leads to an unsafe pair.

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