THE CYCLE OF SIX

LOUIS W. SHAPIRO
Howard University, Washington, D.C. 20059

ABSTRACT

The purpose of this paper is to show that a certain automorphism has order six when restricted to compositions considered as plane trees.

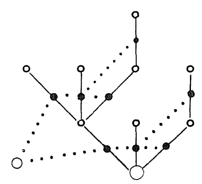
Part I is devoted to the proof of this and in Part II some applications are given. In particular, a duality between various Fibonacci families is discussed which also yields some interesting new settings for the Fibonacci families. Some open questions are mentioned in Part III.

The author would like to thank both Bertrand Harper and Robert Donaghey for helpful conversations.

PART I

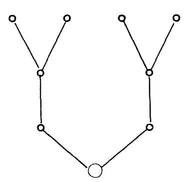
It is well known that plane trees with n edges are equinumerous with binary plane trees with $n+1$ end points. This correspondence was given in a paper by DeBruijn and Morselt [1] in 1967. A modification yields an automorphism on the set of plane trees. Throughout this paper, plane trees will be called trees.

We illustrate this automorphism, which we will denote A, as follows:



, T.

Straightening out the dotted lines yields another plane tree:



, $A(T)$.

Since both T and $A(T)$ have the same number of edges it follows that the number of distinct trees in the sequence T, $A(T)$, $A^2(T)$, $A^3(T)$, is at most

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

since there are C_n trees with n edges.

We give another illustration in Figure I-1. This particular example is not chosen at random; in fact, it illustrates the cycle of six. In general, it is extremely difficult, given a tree T, to predict the order n such that $A^n(T) = T$. Some work has been done on this problem (see [2]) but the central problem remains untouched. This paper represents the first interesting special case.

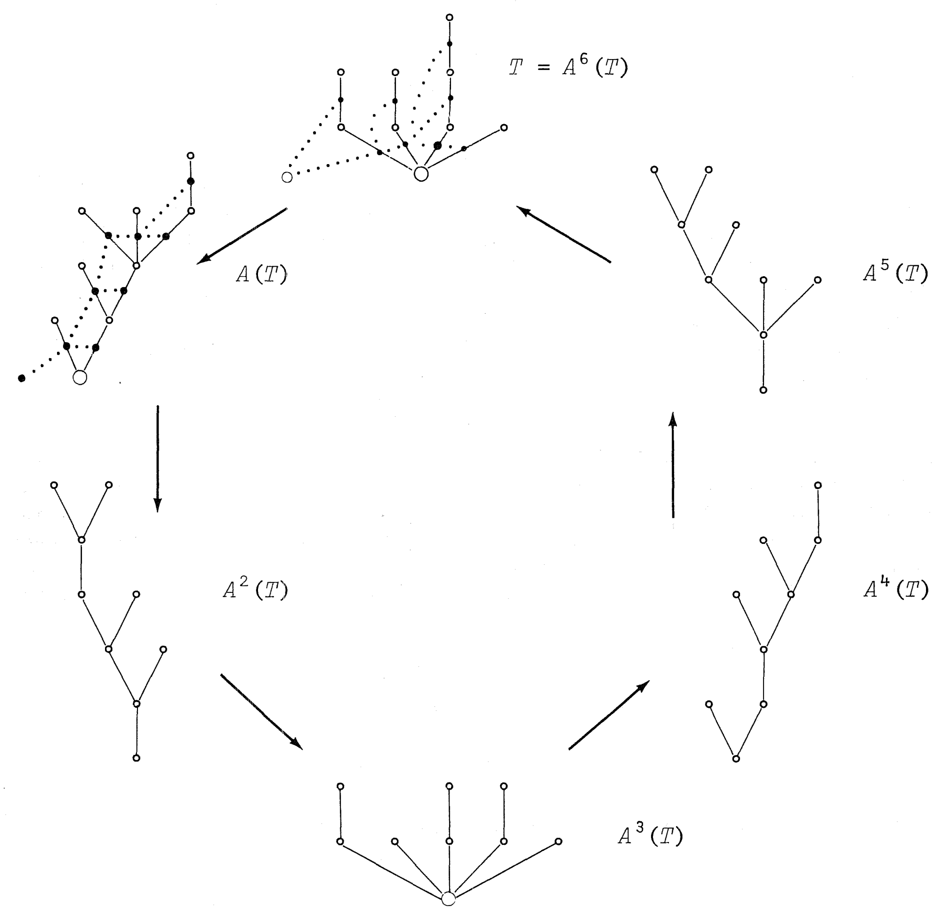
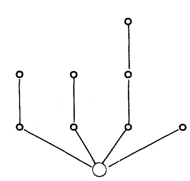


FIGURE I-1

Any composition of a number can easily be represented by a plane tree as follows. If $n = n_1 + n_2 + \cdots + n_k$, then the corresponding tree has only the root as a branch point and the lengths of the branches from the root, going left to right, are $n_1, n_2, \ldots, n_k$. For example,

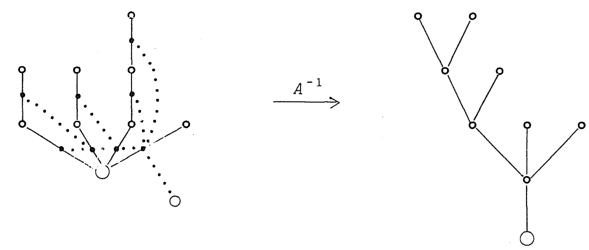
$2 + 2 + 3 + 1 \longleftrightarrow$ 

Theorem: If T represents a composition, then $A^3(T)$ also represents a composition and $A^6(T) = T$.

Proof: We will just trace through the six steps. The illustration is vital for following this proof

Let T be a composition. Since the only branch point is the root, we see that as we are constructing $A(T)$, all of the edges up from a vertex are terminal except the rightmost.

Note next that A^{-1} is defined as is A but from the right. For instance



This shows that this 'terminal-edges-except-for-the-rightmost-edge' condition precisely yields the set $A(T)$ where T is a composition.

Next we have that $A^2(T)$ consists of all trees such that all edges except the leftmost up from a vertex are terminal.

From here it is not hard to see that $A^3(T)$ is again a composition. So, $A^6(T)$ must again be a composition and we only need show $A^6(T) = T$.

Let us define $A^3(T)$ as the dual composition of T.

Suppose $n = n_1 + n_2 + \cdots + n_k$ is the composition that T represents. Then, $A(T)$ has

n_1 edges at the root

n_2 edges at height 2

...

n_k edges at height k.

We construct $A^2(T)$ by first taking a path of length n_1 starting at the root and going up taking the rightmost branch at each node.

Eliminate these n_1 edges and repeat the procedure to get n_2. If elimination disconnects the tree then operate on the upper component first. Continue this procedure to find paths of lengths n_3, n_4, ...

When computing $A^3(T)$, these paths each overlap by 1.

We wish to define a matrix D that will specify the A^3 automorphism exactly. We illustrate this before giving the precise definition:

$$T \longleftrightarrow n = 2 + 2 + 3 + 1 \longleftrightarrow D_T = \begin{array}{cc} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{l} \leftarrow 2 \\ \leftarrow 2 \\ \leftarrow 3 \\ \leftarrow 1 \end{array} \end{array}$$

$A^3(T)$ is given by the column sums read in reverse, here $2 + 1 + 2 + 2 + 1$.

Let $n = n_1 + n_2 + \cdots + n_k$ be a composition T. Then, D_T is a $k \times n - k + 1$ matrix with

$$d_{ij} \quad \begin{cases} 1 & \text{if } \sum_{k=1}^{i-1} n \; - i + 1 \le j \le \sum_{k=1}^{i} n \; - i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$D(2 + 1 + 2 + 2 + 1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is $D(T)$ reflected about the 45° line passing through the middle of the matrix. This situation holds in general.

Repeating this reflection twice yields the original matrix and thus

$$A^6(T) = A^3(A^3(T)) = T,$$

concluding the proof of the theorem.

PART II: SOME APPLICATIONS TO FIBONACCI NUMBERS

The following results were contained in an exercise in a set of lecture notes of R. Stanley.

The following sets are enumerated by the Fibonacci numbers.

A. All compositions of n where all parts are ≥ 2.
B. All compositions of n where all parts are equal to 1 or 2.
C. All compositions of n into odd parts.

These assertions are all easily verified by induction. We will add the following:

D. All compositions, $n = n_1 + n_2 + \cdots + n_{2k+1}$ where all $n_{2j} = 1$.
E. All compositions, $n = n_1 + n_2 + \cdots + n_{2k+1}$ where all $n_j = 1$ for $k + 1 < j \leq 2k$.
F. All compositions, $n = n_1 + n_2 + \cdots + n_m$ where $n_1 \geq n_j$ for $2 \leq j < \ell$, $(-1)^n = (-1)^m$, and $2n_1 + m \geq n + 2$.

Of these, F is perhaps the most interesting. It also seems to be less trivial to prove directly.

For the sake of brevity, we will ignore $A(T)$ and $A^2(T)$ in this discussion and go directly by way of the matrices from T to $A^3(T)$ leaving $A(T)$ and $A^2(T)$ to the diligent reader.

Proposition 1: A and B are dual Fibonacci families (except for a subscript shift).

Let $n = n_1 + n_2 + \cdots + n_k$ where all $n \geq 2$. Then we obtain

$$\begin{array}{c} \begin{array}{cccc} n_1 & n_2 & & n_k \end{array} \\ \begin{pmatrix} 11\ldots1 & & & 0 \\ & 11\ldots1 & & \\ & & \ddots\,11\ldots1 & \\ 0 & & & 11\ldots1 \end{pmatrix} \end{array}$$

The column sums are either 1 or 2 with the first and last column sums always equal to 1. Obviously the compositions of n with first and last parts equal to 1 are bijective with all compositions of $n - 2$. Thus, A and B are essentially dual families, one enumerated by $\{F_n\}$ and the other by $\{F_{n-2}\}$.

We next want to consider the dual of family C. We have

$$n = n_1 + n_2 + \cdots + n_k \text{ where each } n_j \text{ is odd.}$$

For instance

$$3 + 1 + 5 + 1 + 1 + 7 \leftrightarrow \begin{pmatrix} 1 & 1 & 1 & & & & & & & & & & \\ & & 1 & & & & & & & & & & 0 \\ & & 1 & 1 & 1 & 1 & 1 & & & & & & \\ & & & & & & 1 & & & & & & \\ 0 & & & & & & 1 & & & & & & \\ & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The column sums can be larger than 1 only in columns 1, 3, 5, 7, ... This is family D. This time C and D are exact duals and we have proved:

Proposition 2: C and D are dual Fibonacci families.

Proposition 3: E and F are dual Fibonacci families.

Since D and E are equinumerous, E is enumerated by the Fibonacci numbers. We need only show duality. Again we start by looking at an example:

$$11 = n = 1 + 1 + 3 + 2 + 1 + 1 + 1 \quad \text{so that} \quad k = 3,\ 2k + 1 = 7$$

$$\leftrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The last column sum is at least $k + 1$, and this must be as large as any other column sum because each of $n_1, n_2, \ldots, n_{k+1}$ can contribute at most 1 to each column.

Note that the matrix D has $2k + 1$ rows and $n - 2k$ columns. Thus if the dual composition is

$$n = n_1^* + n_2^* + \cdots + n_m^*$$

we have $n_1^* \geq k + 1 = \dfrac{n - m}{2} + 1$, or

$$2n_1^* + m \geq n + 2, \text{ the specification for F.}$$

PART III

To conclude, we mention some open problems and include some related remarks.

1. For a tree T, what is the smallest positive integer k such that $A^k(T) = T$? Even such simple questions as what information about T will guarantee that k is even are unsolved.
2. How many compositions of n with

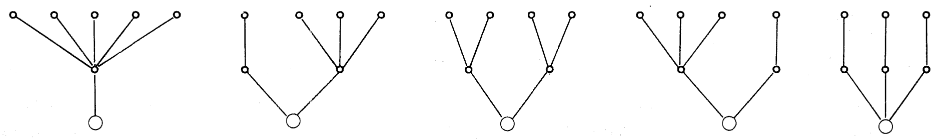
$$n = n_1 + n_2 + \cdots + n_k \text{ have } n_1 \geq n_i \text{ for all } i?$$

A related question would specify also that n and k have the same parity. The first few values are shown in the following table.

n =	1	2	3	4	5	6	7	8	9	10
	1	2	3	5	8	14	24	43	77	130
with $(-1)^{n-k} = 1$	1	1	2	3	5	8	13	22	39	65
with $(-1)^{n-k} = -1$	0	1	1	2	3	6	11	21	38	65

An answer to this question would be of interest in studying partitions.

3. If we specify that all end points of a tree be at height 2 then another Fibonacci family is obtained. For instance, for $n = 6$, we obtain the following five trees:



If we specify height 3 instead of height 2, we obtain the Tribonacci numbers 1, 1, 1, 2, 4, 7, 13, 24, ... If we specify height 3 or less we obtain the sequence 1, 2, 5, 13, 34, 89, ... = $\{F_{2n}\}_{n=0}^{\infty}$. If we knew more about Question 1, we could do more with each of these families. Each of these statements translates into statements about permutations achievable with push down stacks. See Knuth [4] for definitions and explanation.

How many permutations are achievable with a push down stack that holds two elements where each time the stack is empty two elements are put in (or the run ends)? The answer is F_{n-2}, and is equivalent to our first remark in this subsection.

4. What alterations can we make to get reasonably natural settings for the Lucas numbers, the Tribonacci numbers, and the Pell numbers?

One way to obtain the Lucas numbers is to specify compositions

$$n = n_1 + n_2 + \cdots + n_k \text{ where each } n_j \text{ is odd and } n_1 \text{ is 1 or 3.}$$

The dual of this yields the compositions

$$n = n_1 + n_2 + \cdots + n_{2k+1} \text{ with all } n_{2j} = 1 \text{ and } n_1 n_3 \neq 1.$$

5. We have ignored $A(T)$ and $A^2(T)$ throughout. However all the interpretations available for plane trees can be used. See for instance Gardner [3] and the references there. As one example, consider elections where votes are cast one at a time for candidates P and Q. There are $2n$ voters, P never trails Q, and at the end they tie. There are

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

such elections possible. Let us add the condition that the last K votes are for Q but that until then the election was almost monotonic in that if P's lead was ℓ votes, his lead would never be less than $\ell - 1$ thereafter, except for the last K votes. This is just the interpretation of $A(T)$ in Part I. Thus, we see that there are 2^{n-1} such elections, since an integer n has a total of 2^{n-1} compositions.

REFERENCES

1. N. G. DeBruijn & B. J. M. Morselt. "A Note on Plane Trees." *J. Comb. Th.* 2 (1967):27-34.
2. R. Donaghey. Article to appear in *J. Comb. Th.*
3. Martin Gardner. "Mathematical Games." *Scientific American.*
4. D. Knuth. *Fundamental Algorithms.* Vol. I, 2nd ed. Reading: Addison-Wesley, 1973. Pp. 234-239.
5. N. O. Sloane. *A Handbook of Integer Sequences.* New York: Academic Press, 1975.

*****

PROFILE NUMBERS

ARNOLD L. ROSENBERG
IBM Thomas J. Watson Research Center, Yorktown Heights, NY 10598

ABSTRACT

We describe a family of numbers that arises in the study of balanced search trees and that enjoys several properties similar to those of the binomial coefficients.

1. INTRODUCTION

In the course of a recent investigation [4] concerning balanced search trees [2, Section 6.2.3], the following combinatorial problem arose. We encountered in the investigation a family $\{T_L\}$ of $(2L + 1)$-level binary trees, $L = 1, 2, \ldots$; the problem was to determine, as a function of L and $l \in \{0, 1, \ldots, 2L\}$, the number of nonleaf nodes at level l of the $(2L + 1)$-level tree T_L. (By convention, the root of T_L is at level 0, the root's two sons are at level 1, and so on.) The numbers solving this problem, which we call *profile numbers* since, fixing L, the numbers yield the *profile* of the tree T_L [3], that is, the number of nodes at each level of T_L, enjoy a number of features that are strikingly similar to properties of binomial coefficients. Foremost among these similarities are the generating recurrences and summation formulas of the two families of numbers. Let us denote by $P(n,k)$, $n \geq 1$ and $k \geq 0$, the number of nonleaf nodes at level k of the tree T_n, conventionally letting $P(n,k) = 0$ for all $k > 2n$; and let us denote by $C(n,k)$, $n \geq 1$ and $k \geq 0$, the binomial coefficient, conventionally letting $C(n,k) = 0$ for $k > n$. The well-known generating recurrence

$$C(n + 1,k + 1) = C(n,k + 1) + C(n,k), \quad k \geq 0$$

for the binomial coefficients is quite similar to the generating recurrence

$$P(n + 1,k + 1) = P(n,k) + 2P(n,k - 1), \quad k > 0 \tag{1}$$

for profile numbers. Further, the simple closed-form solution of the well-known summation

$$\sum_{0 \leq k < n} C(n,k) = 2^n - 1$$

for binomial coefficients corresponds to the equally simple solution of the