ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate, signed sheets within two months after publication of the problems.

PROBLEMS

H-311 *Proposed by Paul Bruckman, Corcord, CA*

Let a and b be relatively prime positive integers such that ab is not a perfect square. Let $\theta_0 = \sqrt{b/a}$ have the continued fraction expansion

$$[u_1, u_2, u_3, \ldots],$$

with convergents p_n/q_n $(n = 1, 2, \ldots)$; also, define $p_0 = 1$, $q_0 = 0$, $p_{-1} = 0$. The process of finding the sequence $(u_n)_{n=1}^{\infty}$ may be described by the recursions:

$$\theta_n = u_{n+1} + 1/\theta \quad = \frac{\sqrt{ab} + r_n}{d_n}, \tag{1}$$

where $r_0 = 0$, $d_0 = a$, $0 < \theta_n < 1$,

r_n and d_n are positive integers, $n = 1, 2, \ldots$.

Prove:

$$r_n = (-1)^{n-1}(ap_n p_{n-1} - bq_n q_{n-1}); \tag{2}$$

$$d_n = (-1)^n (ap_n^2 - bq_n^2), \; n = 0, 1, 2, \ldots . \tag{3}$$

H-312 *Proposed by L. Carlitz, Duke University, Durham, NC*

Let m, r, and s be nonnegative integers. Show that

$$\sum_{j,k} (-1)^{j+k-r-s}\binom{j}{r}\binom{k}{s}\frac{m!}{(m-j)!\,(m-k)!\,(j+k-m)!} = (-1)^{m-r}\binom{m}{r}\delta_{rs}, \tag{*}$$

where

$$\delta_{rs} = \begin{cases} 1 & (r = s) \\ 0 & (r \neq s). \end{cases}$$

SOLUTIONS

Who's Who?

H-281 *Proposed by V. E. Hoggatt, Jr., San Jose State Univ., San Jose, CA (Vol. 16, No. 2, April 1978)*

Consider the matrix equation:

(a) $$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^n = \begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & G_n \\ H_n & I_n & J_n \end{pmatrix} \quad (n \geq 1).$$

Identify $A_n, B_n, C_n, \ldots, J_n$.

Consider the matrix equation:

(b) $$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} A_n' & B_n' & C_n' \\ D_n' & E_n' & G_n' \\ H_n' & I_n' & J_n' \end{pmatrix} \quad (n \geq 1).$$

Identify $A_n', B_n', C_n', \ldots, J_n'$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

(a) Let
$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

From the symmetry and other properties of A, it follows that A^n has the form
$$\begin{pmatrix} a+1 & b & a \\ b & 2a+1 & b \\ a & b & a+1 \end{pmatrix}.$$

Hence, A^n is determined when the 1st row of A^n is known; also if the 1st row of A^j is (x,y,z), then the 1st row of A^{j+1} will be $(x+y, x+y+z, y+z)$.

For the first five *odd* positive integers, we have the following entries in the 1st row:

Left-hand Entry $u_k + 1$	Middle Entry v_k	Right-hand Entry u_k
1	1	0
4	5	3
21	29	20
120	169	119
697	985	696

We observe that in each case

$$u_k^2 + (u_k + 1)^2 = v_k^2,$$

so $(u_k, u_k + 1, v_k)$ is a (primitive) Pythagorean triple. It is known that Pythagorean triples of the type indicated above are given by

$$u_{k+1} = 6u_k - u_{k-1} + 2 \quad \text{where } u_1 = 0 \text{ and } u_2 = 3$$

and

$$v_{k+1} = 6v_k - v_{k-1} \quad \text{where } v_1 = 1 \text{ and } v_2 = 5.$$

[See Osborne, "A Problem in Number Theory," *Amer. Math. Monthly* (May 1914): 148-150.]

It follows that

$$u_k = \frac{(2 + \sqrt{2})(3 + 2\sqrt{2})^{k-1} - (2 - \sqrt{2})(3 - 2\sqrt{2})^{k-1}}{4\sqrt{2}} - \frac{1}{2},$$
$$k = 1, 2, 3, \ldots .$$

and

$$v_k = \frac{(1 + \sqrt{2})(3 + 2\sqrt{2})^{k-1} - (1 - \sqrt{2})(3 - 2\sqrt{2})^{k-1}}{2\sqrt{2}},$$
$$k = 1, 2, 3, \ldots .$$

[See Example 3-5, pp. 66-67, of Liu, *Introduction to Combinatorial Mathematics* (New York: McGraw-Hill Book Company, 1968), for the procedure used to obtain the above formulas.]

Therefore, for $n \geq 1$, if $n = 2k - 1$, then

$$\begin{array}{lll} A_n = u_k + 1 & B_n = v_k & C_n = u_k \\ D_n = v_k & E_n = 2u_k + 1 & G_n = v_k \\ H_n = u_k & I_n = v_k & J_n = u_k + 1 \end{array}$$

while, if $n = 2k$, then

$$\begin{array}{lll} A_n = u_k + v_k + 1 & B_n = u_k + v_k + 1 & C_n = u_k + v_k \\ D_n = 2u_k + v_k + 1 & E_n = 2u_k + 2v_k + 1 & G_n = 2u_k + v_k + 1 \\ H_n = u_k + v_k & I_n = 2u_k + v_k + 1 & J_n = u_k + v_k + 1 \end{array}$$

It is interesting to note that, for n even, the entry in the upper right-hand corner of A^n is the subscript of a triangular number that is a perfect square. [Recall that $t_1 = 1^2$, $t_8 = 6^2$, $t_{49} = 35^2$, $t_{288} = 204^2$, $t_{1681} = 1189^2$, $t_{9800} = 6930^2$, etc.]

(b) Let

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

From the symmetry and other properties of B, it follows that:

$$B^{2k-1} = \begin{pmatrix} 0 & 2^{k-1} & 0 \\ 2^{k-1} & 0 & 2^{k-1} \\ 0 & 2^{k-1} & 0 \end{pmatrix}, \quad k = 1, 2, 3, \ldots,$$

and

$$B^{2k} = \begin{pmatrix} 2^{k-1} & 0 & 2^{k-1} \\ 0 & 2 & 0 \\ 2^{k-1} & 0 & 2^{k-1} \end{pmatrix}, \quad k = 1, 2, 3, \ldots .$$

This can easily be verified in each of the two cases indicated above using induction and the fact that

$$B^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

[Just multiply B^{2k-1} by B^2 and multiply B^{2k} by B^2.]

Therefore, for $n \geq 1$, if $n = 2k - 1$, then

$$\begin{array}{lll} A_n' = 0 & B_n' = 2^{k-1} & C_n' = 0 \\ D_n' = 2^{k-1} & E_n' = 0 & G_n' = 2^{k-1} \\ H_n' = 0 & I_n' = 2^{k-1} & J_n' = 0 \end{array}$$

while, if $n = 2k$, then

$$\begin{array}{lll} A_n' = 2^{k-1} & B_n' = 0 & C_n' = 2^{k-1} \\ D_n' = 0 & E_n' = 2^{k-1} & G_n' = 0 \\ H_n' = 2^{k-1} & I_n' = 0 & J_n' = 2^{k-1} \end{array}$$

Also solved by P. Bruckman, G. Wulczyn, R. Giuli, and P. Russell.

Speedy Series

H-282 *Proposed by H. W. Gould and W. E. Greig, West Virginia University (Vol. 16, No. 2, April 1978)*

Prove

$$\sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n} - 1} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{a^{2k} - 1},$$

where $a = (1 + \sqrt{5})/2$, and determine which series converges the faster.

Solution by Robert M. Giuli, San Jose State University, San Jose, CA

The equivalence of the two relations is easily established algebraically if $a^{4n} \neq 1$, and disregarding convergence,

$$\sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n} - 1} = \sum_{n=1}^{\infty} \frac{-a^{2n}}{1 - a^{4n}} = \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} - a^{2n} a^{4nr} = \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} - a^{(2n)(2r+1)}$$

Or if $k = 2r + 1$ ($k = 1, 3, 5, \ldots$) and $a^{2n} \neq 1$,

$$\sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n} - 1} = -\sum_{k} \sum_{n=1}^{\infty} a^{2nk} = \sum_{k} \frac{-1}{1 - a^{2k}} = \sum_{k} \frac{1}{a^{2k} - 1}.$$

To show "speed" of convergence, the relation may be rewritten as

$$\sum_{n=1}^{\infty} \frac{1}{a^{2n} - a^{-2n}} = \sum_{n=1}^{\infty} \frac{1}{a^{4n-2} - 1}$$

where $k = 2n - 1$. The series whose terms decrease in magnitude the fastest will converge the fastest (noting that all terms are positive for $a = 1.618$). For $n = 1, 2, 3, \ldots$, we conjecture then that

$$\frac{1}{a^{2n} - a^{-2n}} > \frac{1}{a^{4n-2} - 1} \quad \text{or that } a^{2n} - a^{-2n} < a^{4n-2} - 1,$$

which is easily established, by induction, to be true for $n = 2, 3, 4, \ldots$. Therefore, the right-hand side series of odd terms converges the fastest.

Also solved by L. Carlitz, P. Bruckman, and E. Robinson.

Close Ranks!

H-283 *Proposed by D. Beverage, San Diego Evening College, San Diego, CA (Vol. 16, No. 2, April 1978)*

Define $f(n)$ as follows:

$$f(n) = \sum_{k=0}^{n} \binom{n+k}{n} \left(\frac{1}{2}\right)^{n+k} \qquad (n \geq 0).$$

Express $f(n)$ in closed form.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

We shall show that $f(n) = 1$. Since $f(0) = 1$, to complete our proof, it suffices to show that $f(n + 1) - f(n) = 0$ for $n = 0, 1, 2, \ldots$. Now,

$$f(n+1) - f(n) = \sum_{k=0}^{n+1} \binom{n+1+k}{n+1} \left(\frac{1}{2}\right)^{n+1+k} - \sum_{k=0}^{n} \binom{n+k}{n} \left(\frac{1}{2}\right)^{n+k}$$

$$= \binom{2n+2}{n+1} \left(\frac{1}{2}\right)^{2n+2} + \sum_{k=0}^{n} \left[\binom{n+1+k}{n+1} \left(\frac{1}{2}\right)^{n+1+k} - \binom{n+k}{n} \left(\frac{1}{2}\right)^{n+k}\right]$$

$$= \binom{2n+2}{n+1} \left(\frac{1}{2}\right)^{2n+2} + \sum_{k=0}^{n} \left[\binom{2n+1-k}{n+1} \left(\frac{1}{2}\right)^{2n+1-k} - \binom{2n-k}{n} \left(\frac{1}{2}\right)^{2n-k}\right].$$

Let

$$S_j = \sum_{k=0}^{j} \left[\binom{2n+1-k}{n+1}\left(\frac{1}{2}\right)^{2n+1-k} - \binom{2n-k}{n}\left(\frac{1}{2}\right)^{2n-k}\right]$$

and

$$s_j = \binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{2n+2} + S_j .$$

Claim:

$$s_j = \binom{2n-j}{n+1}\left(\frac{1}{2}\right)^{2n-j}.$$

We have,

$$\begin{aligned}
s_0 &= \binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{2n+2} + \left[\binom{2n+1}{n+1}\left(\frac{1}{2}\right)^{2n+1} - \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right] \\
&= \binom{2n+1}{n+1}\left(\frac{1}{2}\right)^{2n+1} + \left[\binom{2n+1}{n+1}\left(\frac{1}{2}\right)^{2n+1} - \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right] \\
&= \left[\binom{2n+1}{n+1} - \binom{2n}{n}\right]\left(\frac{1}{2}\right)^{2n} \\
&= \left[\binom{2n}{n+1} + \binom{2n}{n} - \binom{2n}{n}\right]\left(\frac{1}{2}\right)^{2n} \\
&= \binom{2n}{n+1}\left(\frac{1}{2}\right)^{2n}
\end{aligned}$$

so the desired result holds when $j = 0$. Assume that

$$s_t = \binom{2n-t}{n+1}\left(\frac{1}{2}\right)^{2n-t}.$$

Then

$$\begin{aligned}
s_{t+1} &= s_t + \binom{2n-t}{n+1}\left(\frac{1}{2}\right)^{2n-t} - \binom{2n-t-1}{n}\left(\frac{1}{2}\right)^{2n-t-1} \\
&= \binom{2n-t}{n+1}\left(\frac{1}{2}\right)^{2n-t-1} - \binom{2n-t-1}{n}\left(\frac{1}{2}\right)^{2n-t-1} \\
&= \left[\binom{2n-t-1}{n+1} + \binom{2n-t-1}{n} - \binom{2n-t-1}{n}\right]\left(\frac{1}{2}\right)^{2n-t-1} \\
&= \binom{2n-t-1}{n+1}\left(\frac{1}{2}\right)^{2n-t-1}.
\end{aligned}$$

The claimed result now follows.

Finally,

$$f(n+1) - f(n) = s_{n-1} + \binom{n+1}{n+1}\left(\frac{1}{2}\right)^{n+1} - \binom{n}{n}\left(\frac{1}{2}\right)^{n}$$

$$= \binom{n+1}{n+1}\left(\frac{1}{2}\right)^{n+1} + \binom{n+1}{n+1}\left(\frac{1}{2}\right)^{n+1} - \binom{n}{n}\left(\frac{1}{2}\right)^{n}$$

$$= \left(\frac{1}{2}\right)^{n} - \left(\frac{1}{2}\right)^{n} = 0.$$

An interesting corollary to the result of this problem is that

$$\sum_{k=0}^{n} \binom{n+k}{n}\left(\frac{1}{2}\right)^{k} = 2^{n}.$$

NOTE: It is also true that

$$f(n+1) - f(n) = s_n = \binom{n}{n+1}\left(\frac{1}{2}\right)^{n} = 0$$

by the usual conventions employed with binomial coefficients because

$n < n + 1$.

Also solved by L. Carlitz, W. Moser, P. Bruckman, and P. Russell.

Late Acknowledgments

H-278 *Also solved by J. Shallit.*

H-279 *Also solved by G. Lord.*

H-280 *Also solved by G. Lord, L, Carlitz, and B. Prielipp.*

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