The proof of Theorem 10 is similar to the proof of Theorem 6, except that one needs Theorem 1 to show that $S_3^*(n) = S_2^*(S_2^*(n))$. The rest of the proof is omitted.

An immediate consequence of Theorem 10 is that if we omit the column when $n = 0$, then every row is a subset of every row preceding it. That is,

$$\{S_1^*(n)\} \supseteq \{S_2^*(n)\} \supseteq \{S_3^*(n)\} \supseteq \{S_4^*(n)\} \supseteq \{S_5^*(n)\} \ \ldots, \tag{17}$$

provided $n \neq 0$.

Using an inductive argument similar to that of Theorem 7, one can show

Theorem 11: If $m \geq 1$ is an integer and $n \neq 0$,

$$S_2^*(S_m^*(n)) = S_m^*(S_2^*(n)).$$

Combining Theorems 10 and 11, we have

$$S_2^*(S_m^*(n)) = S_m^*(S_2^*(n)) = S_{m+1}^*(n) = S_1^*(S_{m+1}^*(n)), \ n \neq 0, \tag{18}$$

and

$$S_3^*(S_m^*(n)) = S_2^*\big(S_2^*(S_m^*(n))\big) = S_2^*\big(S_m^*(S_2^*(n))\big) = S_2^*(S_{m+1}^*(n)), \ n \neq 0. \tag{19}$$

Together, (18) and (19) yield

$$C^*_{S^*_{m+1}(n)} \subseteq C^*_{S^*_m(n)} \tag{20}$$

for all integers $m \geq 1$, $n \neq 0$, where C_i^* is the ith column of Table 2.

The next result, whose proof we omit, since it is by mathematical induction, establishes a relationship between Table 1 and Table 2.

Theorem 12: If m is an integer, $m \geq 1$, $n \neq 0$, then $S_m^*(n) = S_m(n) - F_m + 1$.

Using the fact that $S_2^*(n) = S_2(n)$ in Theorem 10 and applying Theorem 12, we have

$$S_{m+1}(n) + 1 - F_{m+1} = S_{m+1}^*(n) = S_m^*(S_2^*(n)) = S_m(S_2(n)) - F_m + 1$$

or

$$S_{m+1}(n) = S_m(S_2(n)) + F_{m-1}, \ n \neq 0. \tag{21}$$

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1. V. E. Hoggatt, Jr. & A. P. Hillman. "A Property of Wythoff Pairs." *The Fibonacci Quarterly* 16, No. 5 (1978):472.
2. Ivan Niven. *Diophantine Approximations.* Tracts in Mathematics, #14. New York: Interscience Publishers of Wiley & Sons, Inc., 1963, p. 34.
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THE APOLLONIUS PROBLEM

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On p. 326 of *The Fibonacci Quarterly* 12, No. 4 (1974), Charles W. Trigg gave a formula for the radius of a circle which touches three given circles which, in turn, touch each other externally.

The following is a more general formula:

Given triangle ABC with AB = α, BC = β, CA = γ, and circles with centres A, B, and C having radii a, b, and c, respectively.

Let $\ell = a + b + \alpha$; $m = b + c + \beta$; $n = \alpha + b - a$; $p = \beta + b - c$; $q = a + b - \alpha$; $t = b + c - \beta$; $u = \alpha + a - b$; $v = \beta + c - b$; $s = (\alpha + \beta + \gamma)/2$.

Then, if x is the radius of a circle touching the three given ones:

$$4(x + b)\sqrt{s(s - \gamma)} = \sqrt{np(2x + \ell)(2x + m)} \pm \sqrt{uv(2x + q)(2x + t)}$$

the positive sign being taken if the centre of the required circle falls outside angle ABC, and the negative sign if it falls inside angle ABC.

The formula applies to *external* contact. If a given circle of radius a, say, is to make *internal* contact with the required one, then $-a$ must replace $+a$ in the formula. If a given circle of radius a, say, becomes a point, put $a = 0$.

When the three given circles touch each other externally,

$$\alpha = a + b,\ \beta = b + c, \text{ and } \gamma = a + c,$$

and the above formula yields the solution mentioned by Trigg, viz.

$$x = abc/[2\sqrt{abc(a + b + c)} \pm (ab + bc + ca)].$$

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LETTER TO THE EDITOR

L. A. G. DRESEL
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Dear Professor Hoggatt,

In a recent article with Claudia Smith [*Fibonacci Quarterly* 14 (1976): 343], you referred to the question whether a prime p and its square p^2 can have the same rank of apparition in the Fibonacci sequence, and mentioned that Wall (1960) had tested primes up to 10,000 and not found any with this property.

I have recently extended this search and found that no prime up to one million (1,000,000) has this property.

My computations in fact test the Lucas sequence for the property

(1) $$L_p \equiv 1 \pmod{p^2} \qquad p = \text{prime}.$$

For $p > 5$, this is easily shown to be a necessary and sufficient condition for p and p^2 to have the same rank of apparition in the Fibonacci sequence, because of the identity

(2) $$(L_p - 1)(L_p + 1) = 5F_{p-1}F_{p+1}.$$

So far, I have shown that the congruence (1) does not hold for any prime less than one million; I hope to extend the search further at a later date.

You may wish to publish these results in *The Fibonacci Quarterly*.

Yours sincerely,

[*Dr L. A. G. Dresel*]

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