ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-418 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $n^{15} - n^3$ is an integral multiple of $2^{15} - 2^3$ for all integers n.

B-419 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

For i in $\{1, 2, 3, 4\}$, establish a congruence

$$F_n L_{5k+i} \equiv a_i n L_n F_{5k+i} \pmod 5$$

with each a_i in $\{1, 2, 3, 4\}$.

B-420 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Let

$$g(n, k) = F^4_{n+10k} + F^4_n - (L_{4k} + 1)(F^4_{n+8k} + F^4_{n+2k}) + L_{4k}(F^4_{n+6k} + F^4_{n+4k}).$$

Can one express $g(n, k)$ in the form $L_r F_s F_t F_u F_v$ with each of r, s, t, u, and v linear in n and k?

B-421 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $\{u_n\}$ be defined by the recursion $u_{n+3} = u_{n+2} + u_n$ and the initial conditions $u_1 = 1$, $u_2 = 2$, and $u_3 = 3$. Prove that every positive integer N has a unique representation

$$N = \sum_{i=1}^{n} c_i u_i,\ \text{with } c_n = 1,\ \text{each } c_i \in \{0, 1\},$$

$$c_i c_{i+1} = 0 = c_i c_{i+2} \quad \text{if} \quad 1 \le i \le n - 2.$$

B-422 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

With representations as in B-421, let

$$N = \sum_{i=1}^{n} c_i u_i,\ N + 1 = \sum_{i=1}^{m} d_i u_i.$$

Show that $m \geq n$ and that if $m = n$ then $d_k > c_k$ for the largest k with

$$c_k \neq d_k.$$

B-423 *Proposed by Jeffery Shallit, Palo Alto, CA*

Here let F_n be denoted by $F(n)$. Evaluate the infinite product

$$\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{13}\right)\left(1 + \frac{1}{610}\right) \cdots = \prod_{n=1}^{\infty}\left[1 + \frac{1}{F(2^{n+1} - 1)}\right].$$

SOLUTIONS

Note by Paul S. Bruckman, Concord, CA:

There is an omission in the published solution to B-371 (Feb. 1979, p. 91). The set of residues (mod 60) should include 55 and consists of 24 elements.

Triple Products and Binomial Coefficients

B-394 *Proposed by Phil Mana, Albuquerque, NM*

Let $P(x) = x(x - 1)(x - 2)/6$. Simplify the following expression:

$$P(x + y + z) - P(y + z) - P(x + z) - P(x + y) + P(x) + P(y) + P(z).$$

I. *Solution by C. B. A. Peck, State College, PA*

Let $G(x, y, z)$ denote the given expression. Clearly,

$$G(0, y, z) = G(x, 0, z) = G(x, y, 0) = 0.$$

Since the total degree of G in x, y, z is at most 3, this means that

$$G = kxyz, \text{ with } k \text{ constant.}$$

Then $G(1, 1, 1) = 1$ implies that $k = 1$ and $G = xyz$.

II. *Generalization by L. Carlitz, Duke University, Durham, NC*

We shall prove the following more general result. Let

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

be an arbitrary polynomial of degree $\leq n$ and put

$$\begin{aligned} S_n &= S_n(x_1, x_2, \ldots, x_n) \\ &= f(x_1 + x_2 + \cdots + x_n) - \Sigma f(x_1 + \cdots + x_{n-1}) \\ &\quad + \Sigma f(x_1 + \cdots + x_{n-2}) - \cdots + (-1)^n f(0), \end{aligned}$$

where the first sum is over all sums of $n - 1$ of the x_j, the second over all sums of $n - 2$ of the x_j, etc. Then

$$(*) \qquad S_n = a_0 n! x_1 x_2 \cdots x_n.$$

Proof: Put

$$S(n,k;x_1, \ldots, x_k) = (x_1 + \cdots + x_k)^n - \Sigma(x_1 + \cdots + x_{k-1})^n + \Sigma(x_1 + \cdots + x_{k-2})^n - \cdots,$$

where the summations have the same meaning as in the definition of S_n, except that we now have k indeterminates.

It follows from the definition that

$$(**) \qquad \sum_{n=0}^{\infty} S(n,k;x_1, \ldots, x_k)\frac{z}{n!}$$

$$= e^{(x_1 + \cdots + x_k)z} - \Sigma e^{(x_1 + \cdots + x_{k-1})z} + \Sigma e^{(x_1 + \cdots + x_{k-2})z}$$

$$= (e^{x_1 z} - 1)(e^{x_2 z} - 1) \cdots (e^{x_k z} - 1).$$

Hence, comparing coefficients of z^n, we get

$$S(n,k;x_1, \ldots, x_k) = \begin{cases} 0 & (0 \leq n < k) \\ k! x_1 \cdots x_k & (n = k) \end{cases}$$

The assertion (*) is an immediate consequence.

For $n = 3$, $a = \frac{1}{6}$, (*) reduces to the required result.

Remark: For $x_1 = \cdots = x_k = 1$, it is clear that (**) reduces to

$$\sum_{n=0}^{\infty} S(n,k;1, \ldots, 1)\frac{z^n}{n!} = (e^z - 1)^k.$$

Hence,

$$S(n,k;1, \ldots, 1) = k! S(n,k),$$

where

$$S(n,k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n,$$

a Stirling number of the second kind.

Also solved by Mangho Ahuja, Paul S. Bruckman, Herta T. Freitag, Graham Lord, John W. Milsom, Charles B. Shields, Sahib Singh, Gregory Wulczyn, and the proposer.

Reciprocals of Golden Powers

B-395 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $c = (\sqrt{5} - 1)/2$. For $n = 1, 2, \ldots$, prove that

$$1/F_{n+2} < c^n < 1/F_{n+1}.$$

Solution by Sahib Singh, Clarion State College, Clarion, PA

Since $c = \frac{1}{a}$, it suffices to show that $F_{n+2} > a^n > F_{n+1}$. Consider

$$F_{n+2} - a^n = \frac{a^{n+2} - b^{n+2}}{a - b} - a^n(a + b) = b^2 F_n.$$

Similarly, $a^n - F_{n+1} = -bF_n$.

Since b is negative, the conclusion follows.

Also solved by Mangho Ahuja, Clyde A. Bridger, Paul S. Bruckman, Herta T. Freitag, Graham Lord, C. B. A. Peck, Bob Prielipp, E. D. Robinson, Charles B. Shields, Lawrence Somer, and the proposer.

Multiples of Ten

B-396 *Based on the solution to B-371 by Paul S. Bruckman, Concord, CA*

Let $G_n = F_n(F_n + 1)(F_n + 2)(F_n + 3)/24$. Prove that 60 is the smallest positive integer m such that $10|G_n$ implies $10|G_{n+m}$.

Solution by Paul S. Bruckman, Concord, CA

In B-371, it was shown that $10|G_n$ iff $n \equiv r \pmod{60}$, where r is any of 24 possible given residues (mod 60). Thus,

$$n \equiv r \pmod{60} \Longleftrightarrow 10|G_n \Rightarrow 10|G_n \Longleftrightarrow n + m \equiv r \pmod{60},$$

or, equivalently,

$$n \equiv r \pmod{60} \Rightarrow n + m \equiv r \pmod{60} \Rightarrow m \equiv 0 \pmod{60} \Rightarrow 60\ m.$$

Clearly, the smallest m with this property is $m = 60$, since *any* multiple of 60 (including 60 itself) has the property. See note after B-423.

Also solved by C. B. A. Peck, Sahib Singh, Lawrence Somer, & Gregory Wulczyn.

Semi-Closed Form

B-397 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Find a closed form for the sum

$$\sum_{k=0}^{2s} \binom{2s}{k} F^2_{n+kt}.$$

Note: The proposer intended t to be odd but this condition was inadvertently omitted by the elementary problems editor. The solution which follows gives a closed form for t even and for t odd.

Solution by Paul S. Bruckman, Concord, CA

Let

$$\theta_{2s,n,t} = \sum_{k=0}^{2s} \binom{2s}{k} F^2_{n+kt}.$$

Then

$$\theta_{2s,n,t} = \frac{1}{5} \sum_{k=0}^{2s} \binom{2s}{k} \left\{a^{2n+2kt} - 2(-1)^{n+kt} + b^{2n+2kt}\right\}$$

(continued)

$$= \frac{1}{5}\left\{a^{2n}(1 + a^{2t})^{2s} - 2(-1)^n\{1 + (-1)^t\}^{2s} + b^{2n}(1 + b^{2t})^{2s}\right\},$$

or

(1) $$\theta_{2s,n,t} = \frac{1}{5}\left\{a^{2n+2st}(a^t + a^{-t})^{2s} + b^{2n+2st}(b^t + b^{-t})^{2s} - 2^{2s+1}(-1)^n\left(\frac{1 + (-1)^t}{2}\right)\right\}$$

We may distinguish two cases, in order to further simplify (1):

$$\theta_{2s,n,2u} = \frac{1}{5}\{(a^{2n+4su} + b^{2n+4su})(a^{2u} + b^{2u})^{2s} - (-1)^n 2^{2s+1}\},$$

or

(2) $$\theta_{2s,n,2u} = \frac{1}{5}(L_{2n+4su}L_{2u}^{2s} - (-1)^n 2^{2s+1});$$

also

$$\theta_{2s,n,2u+1} = \frac{1}{5}(a^{2n+2s(2u+1)} + b^{2n+2s(2u+1)})(a^{2u+1} - b^{2u+1})^{2s},$$

or

(3) $$\theta_{2s,n,2u+1} = 5^{s-1}L_{2n+2s(2u+1)}F_{2u+1}^{2s}.$$

Also solved for t odd by the proposer.

The Added Ingredient

B-398 *Proposed by Herta T. Freitag, Roanoke, VA*

Is there an integer K such that

$$K - F_{n+6} + \sum_{j=1}^{n} j^2 F_j$$

is an integral multiple of n for all positive integers n?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

According to (17) on p. 215 of the October 1965 issue of this journal,

$$\sum_{k=0}^{n} k^2 F_k = (n^2 + 2)F_{n+2} - (2n - 3)F_{n+3} - 8.$$

Since $F_{n+6} = 3F_{n+3} + 2F_{n+2}$, it follows that

$$8 - F_{n+6} + \sum_{j=1}^{n} j^2 F_j = n(nF_{n+2} - 2F_{n+3})$$

where n is an arbitrary positive integer.

Also solved by Paul S. Bruckman, Sahib Singh, Gregory Wulczyn, and the proposer.

Not Quite Tribonacci

B-399 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $f(x) = u_1 + u_2x + u_3x^2 + \cdots$ and $g(x) = v_1 + v_2x + v_3x^2 + \cdots$ where $u_1 = u_2 = 1$, $u_3 = 2$, $u_{n+3} = u_{n+2} + u_{n+1} + u_n$, and $v_{n+3} = v_{n+2} + v_{n+1} + v_n$. Find initial values v_1, v_2, and v_3 so that $e^{g(x)} = f(x)$.

I. *No such series exists.*
Demonstration by Jonathan Weitsman, College Station, TX

The equation $e^{g(x)} = f(x)$ leads to $v_1 = 0$, $v_2 = 1$, $v_3 = 3/2$, and $v_4 = 7/3$. These values contradict the given recursion for the v's.

II. *Correction and solution by Paul S. Bruckman, Concord, CA*

There is an error in the statement of the problem. One correct rewording would be to replace "$g(x)$" where it *first* occurs by "$g'(x)$".

Note that $u_n = T_{n+2}$, where $(T_n)_{n=0}^{\infty} = (0, 0, 1, 1, 2, 4, 7, 13, 24, \ldots)$ is the Tribonacci sequence; also $f(x) = (1 - x - x^2 - x^3)^{-1}$, the well-known generating function for the Tribonacci numbers.

Since the v_n's satisfy the same recursion, it follows that

$$g'(x) = p(x)f(x),$$

where p is some quadratic polynomial. But, if we are to have

$$f(x) = \exp(g(x)),$$

then $g(x) = \log(f(x))$, and $g'(x) = f'(x)/f(x)$. Hence,

$$p(x) = f'(x)/f^2(x) = -\frac{d}{dx}(1/f(x)) = -\frac{d}{dx}(1 - x - x^2 - x^3) = 1 + 2x + 3x^2.$$

Therefore,

$$g'(x) = \frac{1 + 2x + 3x^2}{1 - x - x^2 - x^3} \quad \text{and} \quad g(x) = c - \log(1 - x - x^2 - x^3),$$

for some constant c. Since $g(0) = \log(f(0)) = \log 1 = 0 = c - \log 1 = c - 0 = c$, thus $g(x) = -\log(1 - x - x^2 - x^3)$.

Now $g'(x) = p(x)f(x) = (1 + 2x + 3x^2)\sum_{n=0}^{\infty} T_{n+2}x^n$; hence,

$$g'(x) = \sum_{n=0}^{\infty} T_{n+2}x^n + 2\sum_{n=1}^{\infty} T_{n+1}x^n + 3\sum_{n=2}^{\infty} T_n x^n$$

$$= \sum_{n=0}^{\infty} (T_{n+2} + 2T_{n+1} + 3T_n)x^n = \sum_{n=0}^{\infty} v_{n+1}x^n,$$

which implies $v_{n+1} = T_{n+2} + 2T_{n+1} + 3T_n$, $n = 0, 1, 2, \ldots$. Thus, $v_1 = 1 + 0 + 0 = 1$; $v_2 = 1 + 2 + 0 = 3$; and $v_3 = 2 + 2 + 3 = 7$. The first few terms of the series for $g(x)$ are as follows:

$$g(x) = -\log(1 - x - x^2 - x^3) = \frac{1x}{1} + \frac{3x^2}{2} + \frac{7x^3}{3} + \frac{11x^4}{4} + \frac{21x^5}{5} + \frac{39x^6}{6} + \cdots .$$

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