When $k = 3$, then $F_{m-3} + F_{m+3} = 2H_m$. Now $2|H_m$ if and only if $3|m$. Thus, if $(2, H_m) = 1$ we have that

$$\phi(F_{m+3} + F_{m-3}) = \begin{cases} 12m \text{ if } m \text{ is even} \\ 6m \text{ if } m \text{ is odd.} \end{cases}$$

Finally, it may be worthwhile commenting on the conditions of the form $(H_a, F_b) = 1$ which have been necessary for our computations. $(H_a, F_b) > 1$ is not a rare phenomenon because, for instance, given a it is easy to determine an infinite number of values of b for which $H_a|F_b$. In fact, as we now show, $H_a|F_b$ if and only if b is a positive integral multiple of $2a$. For, $H_a|F_{2a}$ because $F_{2a} = F_a H_a$. Thus, $H_a|F_{2ac}$ for any positive integer c. Actually, $2a$ is the least suffix b for which $H_a|F_b$, as shown by the proof of Theorem B in [2]. Let B denote the set of all positive integers b for which $H_a|F_b$. Then B is nonempty, and if $b_1, b_2 \in B$ since

$$F_{b_1+b_2} = F_{b_1+1}F_{b_2} + F_{b_1}F_{b_2-1}$$
$$F_{b_1-b_2} = (-1)^{b_2}(F_{b_2-1}F_{b_1} - F_{b_2}F_{b_1-1}),$$

we see that $b_1 + b_2$, $b_1 - b_2 \in B$. This means that B consists of all multiples of some least element which, as already pointed out, is $2a$ (see Theorem 6 in Chapter I of [4]).

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MUTUALLY COUNTING SEQUENCES

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ABSTRACT

Let n and m be positive integers with $n \leq m$. Let A be the sequence of n nonnegative integers $a(0), a(1), \ldots, a(n-1)$, and let B be the sequence of m nonnegative integers $b(0), b(1), \ldots, b(m-1)$, where $a(i)$ is the multiplicity of i in B and $b(j)$ is the multiplicity of j in A. We prove that for $n > 7$, there are exactly 3 ways to generate such pairs of sequences.

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Let n and m be positive integers with $n \leq m$. Let A be the sequence of n nonnegative integers $a(0), a(1), \ldots, a(n-1)$, and let B be the sequence of m nonnegative integers $b(0), b(1), \ldots, b(m-1)$, where $a(i)$ is the multiplicity of i in B and $b(j)$ is the multiplicity of j in A. Then A and B

are said to form a pair of *mutually counting sequences*. Observe at the outset that

$$S(A) = \sum_{i=0}^{n-1} a(i) = m \quad \text{and} \quad S(B) = \sum_{j=0}^{m-1} b(j) = n.$$

In this paper, we prove the following result:

Theorem: For $n > 7$, a pair of mutually counting sequences A and B can be formed in exactly 3 ways:

(I) $a(0) = m - 3$, $a(1) = a(3) = a(n - 4) = 1$,
$a(i) = 0$ for all remaining i;
$b(0) = n - 4$, $b(1) = 3$, $b(m - 3) = 1$,
$b(j) = 0$ for all remaining j.

(II) $a(0) = m - 4$, $a(1) = 3$, $a(n - 3) = 1$,
$a(i) = 0$ for all remaining i;
$b(0) = n - 3$, $b(1) = b(3) = b(m - 4) = 1$,
$b(j) = 0$ for all remaining j.

(III) $a(0) = m - 4$, $a(1) = 2$, $a(2) = a(n - 4) = 1$,
$a(i) = 0$ for all remaining i;
$b(0) = n - 4$, $b(1) = 2$, $b(2) = b(m - 4) = 1$,
$b(j) = 0$ for all remaining j.

Proof: Let A and B be a pair of mutually counting sequences. Then clearly, $b(m - 2) + b(m - 1) \leq 1$. Suppose that $b(m - 1) = 1$. Then $m - 1$ has multiplicity 1 in A, and since $S(A) = m$, one of the remaining entries of A must be 1, while the other $n - 2$ entries are 0. Therefore,

$$b(0) = n - 2,\ b(1) = b(m - 1) = 1, \text{ and } b(j) = 0 \text{ for all remaining } j,$$

which implies that $a(1) = 2$, a contradiction. Now suppose that $b(m - 2) = 1$. Then $m - 2$ has multiplicity 1 in A. Again, from $S(A) = m$, we may conclude that either (i) one of the remaining entries of A is 2 and the other $n - 2$ entries are 0 or (ii) two of the remaining entries of A are 1 and the other $n - 3$ entries are 0. In (i), we get

$$b(0) = n - 2,\ b(2) = b(m - 2) = 1,$$

while (ii) yields

$$b(0) = n - 3,\ b(1) = 2,\ b(m - 2) = 1.$$

In both instances it follows from $S(B) = n$ that the remaining $m - 3$ entries of B are 0. But this implies that $a(0) = m - 3$, a contradiction. Thus, we may conclude that the initial inequality must be strict, which immediately gives $b(m - 2) = b(m - 1) = 0$. By an analogous argument, it follows that $a(n - 2) = a(n - 1) = 0$ as well.

Note next that $b(m - 3) \leq 1$. If equality holds, then $m - 3$ has multiplicity 1 in A, and since $S(A) = m$, the sum of the remaining entries of A must be 3. Three possibilities exist for these remaining entries: (i) one is 3 and the other $n - 2$ are 0; (ii) one is 2, another is 1, and the other $n - 3$ are 0; (iii) three are 1 and the other $n - 4$ are 0. In (i), we have $b(0) = n - 2$, contradicting $a(n - 2) = 0$; in (ii), we have

$$b(0) = n - 3,\ b(1) = b(2) = b(m - 3) = 1,$$

and $b(j) = 0$ for all remaining j,

implying that $a(0) = m - 4$, a contradiction; from (iii), we obtain

$$b(0) = n - 4,\ b(1) = 3,\ b(m - 3) = 1,$$
$$\text{and } b(j) = 0 \text{ for all remaining } j,$$

which implies that

$$a(0) = m - 3,\ a(1) = a(3) = a(n - 4) = 1,$$
$$\text{and } a(i) = 0 \text{ for all remaining } i.$$

That is, if $b(m - 3) = 1$, we get a pair of mutually counting sequences of type (I). Similarly, observe that $a(n - 3) \leq 1$. If we assume that $a(n - 3) = 1$, then the same kind of procedure as above will produce a pair of mutually counting sequences of type (II). In what follows, therefore, we will assume without loss of generality that $a(n - 3) = b(m - 3) = 0$.

Now note that $b(m - 4) \leq 1$ [only when $n = m = 8$ is it possible for $b(m - 4) = 2$, and a simple calculation leads to a quick contradiction]. If $b(m - 4) = 1$, then $m - 4$ has multiplicity 1 in A, so that the sum of the remaining entries of A must be 4. There are five possibilities here for these remaining entries: (i) one is 4 and the other $n - 2$ are 0; (ii) one is 3, another is 1, and the other $n - 3$ are 0; (iii) two are 2 and the other $n - 3$ are 0; (iv) four are 1 and the other $n - 5$ are 0; and (v) two are 1, another is 2, and the other $n - 4$ are 0. In (i), we have $b(0) = n - 2$, contradicting $a(n - 2) = 0$; in (ii) and (iii), we get $b(0) = n - 3$, which contradicts $a(n - 3) = 0$; in (iv), we find $b(0) = n - 5$, $b(1) = 4$, and $b(j) = 0$ for all remaining j, which implies that $a(0) = m - 3$, again a contradiction; finally in (v), we have $b(0) = n - 4$, $b(1) = 2$, $b(2) = 1$, and $b(j) = 0$ for all remaining j. This yields

$$a(0) = m - 4,\ a(1) = 2,\ a(2) = 1,\ a(n - 4) = 1,$$
$$\text{and } a(i) = 0 \text{ for all remaining } i.$$

That is, under the stated hypotheses, we have produced a pair of mutually counting sequences of type (III).

It remains to show that if $b(m - 4) = 0$, then no other pair of mutually counting sequences can be constructed. This result is easily verified for $n = 8$ and $n = 9$, so for $n \geq 0$ we will now assume that another such pair exists and will deduce an eventual contradiction.

If $b(j) = 0$ for all $j \geq [m/2]$, then the multiplicity of 0 in B is at least $m - [m/2]$, i.e., $a(0) \geq m - [m/2] \geq [m/2]$. But this implies that some integer $j \geq [m/2]$ appears in A, contradicting the initial assumption. Thus $b(j^*) > 0$ for at least one integer $j^* \geq [m/2]$, where $j^* < m - 4$. If j_1^* and j_2^* are distinct integers with this property, then both appear at least once in A, so that $m = S(A) \geq j_1^* + j_2^* > 2[m/2]$. If m is even, then we obtain $m > m$, which is an obvious contradiction; if m is odd, then $2[m/2] = m - 1$, which gives $j_1^* + j_2^* = S(A)$. It then follows that all remaining entries of A must be 0, so $b(0) = n - 2$. But this contradicts $a(n - 2) = 0$. Therefore, j^* is unique.

Next, it is apparent that $b(j^*) = 1$ or 2. If $b(j^*) = 2$, then we easily conclude that $j^* = [m/2]$, from which it follows that $m = S(A) \geq 2j^* = 2[m/2]$. This again leads to contradictory statements whether m is odd or even, so we may assert that $b(j^*) = 1$.

Suppose that $a(i) = j^*$ for some $i > 2$. Then since the multiplicity of i in B is j^* and the multiplicity of 1 in B is at least 1 [since $b(j^*) = 1$], it follows that $n = S(B) \geq ij^* + 1 > 2[m/2] + 1 \geq m$, a contradiction. If $a(2) = j^*$, then 2 has multiplicity j^* in B, and since $b(0) \geq 3$, we get $n \geq 2j^* + 3 > m$, again a contradiction. Suppose next that $a(1) = j^*$. Since $b(j) = 0$ for all $j \geq [m/2]$, $j \neq j^*$, it follows that $a(0) \geq m - [m/2] - 1$. Therefore,

$$m = S(A) \geq a(0) + a(1) \geq m - [m/2] - 1 + j^* \geq m - 1,$$

which implies that either one of the remaining entries of A is 1 and all others are 0 or all remaining entries of A are 0. So the multiplicity of 0 in A is either $n - 3$ or $n - 2$, implying that either $b(n - 3)$ or $b(n - 2)$ is nonzero, both contradictions. Hence, $a(0) = j^*$.

Now consider the case in which $j^* = [m/2]$. Then $b(j) = 0$ for all $j > [m/2]$, accounting for $m - [m/2] - 1$ entries of 0 in B. Since $a(0) = j^* = [m/2]$, the number of remaining zero entries of B, denoted by r, is given by

$$r = [m/2] - (m - [m/2] - 1) = 2[m/2] - m + 1.$$

If m is odd, then $r = 0$, so in particular, $b([m/2] - k)$, $k = 1, 2, 3$ are all nonzero. This means that in addition to $[m/2]$, the integers $[m/2] - k$, $k = 1, 2, 3$ all appear at least once in A. Then

$$m = S(A) \geq 4[m/2] - 6 = 2m - 8,$$

which yields $m \leq 8$, a contradiction. If m is even, then $r = 1$, so only one of the remaining entries of B is 0. Then at least three of the four entries $b([m/2] - k)$, $k = 1, 2, 3, 4$ are nonzero, which implies that in addition to $[m/2]$, at least three of the integers between $[m/2] - 4$ and $[m/2] - 1$ appear at least once in A. Then

$$m = S(A) \geq [m/2] + ([m/2] - 2) + ([m/2] - 3) + ([m/2] - 4),$$

i.e.,

$$m \geq 4[m/2] - 9 = 2m - 9.$$

But this implies that $m \leq 9$, a contradiction. We conclude that $j^* > [m/2]$.

At this point, we may improve our results concerning the zero entries of B. For, suppose that $b(j) \neq 0$ for some $j \geq m - j^* - 1$, $j \neq j^*$. Then, j and j^* both have multiplicity at least 1 in A, so that

$$m = S(A) \geq j + j^* \geq m - 1.$$

Therefore, either one of the remaining entries of A is 1 and the other $n - 3$ entries are 0, or each of the $n - 2$ remaining entries of A is 0. Then the multiplicity of 0 in A is either $n - 3$ or $n - 2$, implying that either $b(0) = n - 3$ or $b(0) = n - 2$. But this means that either $a(n - 3)$ or $a(n - 2)$ is nonzero, both contradictions. So, $b(j) = 0$ for all $j \geq m - j^* - 1$, $j \neq j^*$, which accounts for precisely j^* entries of 0 in B. Since $a(0) = j^*$, it follows that all remaining entries of B must be nonzero. In particular, $b(m - j^* - 2)$, $b(m - j^* - 3)$, and $b(1)$ are all nonzero, which means that in addition to j^*, the integers $m - j^* - 2$, $m - j^* - 3$, and 1 all appear in A. So

$$m = S(A) \geq (m - j^* - 2) + (m - j^* - 3) + j^* + 1 = 2m - j^* - 4,$$

which implies that $j^* \geq m - 4$, the desired contradiction. Consequently, the assumption that another pair of mutually counting sequences can be generated must be false, and the theorem is proved.

It is left to the interested reader to generate the mutually counting sequences that exist for $n \leq 7$.

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