$\beta_1 = b - 1$, where $b > 1$, and $\beta_{n+1} = \max(\beta_n - 1, 1)$ if $n \geq 1$, we have that $\beta_{b-1} = 1$ if $b \geq 3$, $\beta_2 = 1$ if $b < 3$, and that $\phi^n(3^b) = 2^3 \cdot 3^{\beta_n}$. Thus, there exists an integer $v \geq 2$ such that $\phi^v(3^b) = 2^3 \cdot 3$ if $b > 1$.

Now we note that $\phi^4(2) = \phi^3(3) = 2^3 \cdot 3$ and that $\phi^3(5^c) = 2^3 \cdot 3 \cdot 5^c$ for any $c \geq 1$ and that $\phi(2^3 \cdot 3 \cdot 5^c) = 2^3 \cdot 3 \cdot 5^c$ holds even for $c = 0$. Again using Lemma 14 of [1] we have for $a, b > 1$ that

$$\begin{aligned}\phi^{u+v}(2^a 3^b) &= [\phi^{u+v}(2^a),\ \phi^{u+v}(3^b)]\\ &= [\phi^v(2^3 \cdot 3),\ \phi^u(2^3 \cdot 3)]\\ &= 2^3 \cdot 3,\end{aligned}$$

so that

$$\phi^{u+v}(2^a 3^b 5^c) = [2^3 \cdot 3,\ 2^3 \cdot 3 \cdot 5^c] = 2^3 \cdot 3 \cdot 5^c$$

since $u + v > 3$. Consequently

$$\phi^{u+v+1}(2^a 3^b 5^c) = \phi^{u+v}(2^a 3^b 5^c).$$

The remaining cases are when $a \leq 1$ or $b \leq 1$, and it is easy to check that $\phi^{v+3}(2^a 3^b 5^c) = \phi^{v+2}(2^a 3^b 5^c)$ if $a \leq 1$ and $\phi^{u+3}(2^a 3^b 5^c) = \phi^{u+2}(2^a 3^b 5^c)$ if $b \leq 1$.

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SOME REMARKS ON THE PERIODICITY OF THE SEQUENCE OF FIBONACCI NUMBERS—II

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The Fibonacci sequence $\{F_n\}$ is defined by

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1} \quad (n \geq 1).$$

If t is an integer greater than 2 and $\phi(t)$ is the length of the period of the sequence reduced to least nonnegative residues modulo t, it was shown in [2] that $\phi(F_{m-1} + F_{m+1}) = 4m$ if m is even and $\phi(F_{m-1} + F_{m+1}) = 2m$ if m is odd. It follows for $m > 4$ that

$$\phi(F_{m-1} + F_{m+1}) = \frac{1}{2}(\phi(F_{m-1}) + \phi(F_{m+1})).$$

I conjectured in the same paper that if $m - k > 3$ then

$$\phi(F_{m-k} + F_{m+k}) = \frac{k}{2}(\phi(F_{m-k}) + \phi(F_{m+k})).$$

The object of this note is to show that this conjecture is false and to give the correct answer in some special cases.

That the conjecture is false may be seen by taking $m = 12$ and $k = 4$, for example, because in this case

$$\phi(F_8 + F_{16}) = \phi(1008) = 48,$$

whereas

$$2(\phi(F_8) + \phi(F_{16})) = 96.$$

In what follows, we write $[x,y]$ and (x,y) for the lowest common multiple and the greatest common divisor of the integers x and y, respectively, and let x_2 denote the largest number e for which $2^e | x$. Also we define

$$H_a = F_{a-1} + F_{a+1} \quad (a \geq 1).$$

Theorem: Suppose that k and m are integers with $3 < k \leq m$. Then

(i) if k is even and $(H_m, F_k) = 1$, we have

$$\phi(F_{m-k} + F_{m+k}) = \begin{cases} 2[k,m] \text{ if } m \text{ is even and } k_2 < m_2 \\ 4[k,m] \text{ otherwise,} \end{cases}$$

(ii) if k is odd and $(H_k, F_m) = 1$, we have

$$\phi(F_{m-k} + F_{m+k}) = 4[k,m].$$

The proof of this requires the fact that if $n = \alpha\beta$ and $(\alpha,\beta) = 1$, then $\phi(n) = [\phi(\alpha), \phi(\beta)]$, essentially proved in Theorem 2 of [3]. Now it is well known that

$$F_{m-k} = (-1)^k (F_{k-1}F_m - F_k F_{m-1})$$
$$F_{m+k} = F_{k+1}F_m + F_k F_{m-1},$$

so that

$$F_{m-k} + F_{m+k} = \begin{cases} H_k F_m \text{ if } k \text{ is even} \\ H_m F_k \text{ if } k \text{ is odd.} \end{cases}$$

Consequently, if k is even and $(H_k, F_m) = 1$, then

$$\phi(F_{m-k} + F_{m+k}) = [\phi(H_m), \phi(F_k)] = \begin{cases} [4k, 2m] \text{ if } m \text{ is even} \\ [4k, 4m] \text{ if } m \text{ is odd,} \end{cases}$$

using results proved in [1] and [2]. Similarly, if k is odd and $(H_m, F_k) = 1$, we have that

$$\phi(F_{m-k} + F_{m+k}) = [\phi(H_k), \phi(F_m)] = \begin{cases} [4m, 4k] \text{ if } m \text{ is even} \\ [2m, 4k] \text{ if } m \text{ is odd.} \end{cases}$$

The result now follows by noting that if k and m are even then $[4k, 2m]$ equals $2[k,m]$ or $4[k,m]$ depending on whether $k_2 < m_2$ or $k_2 \geq m_2$, respectively; if k is even and m is odd then $[4k, 4m] = 4[k,m]$, and if k and m are both odd then $[4k, 2m] = 4[k,m]$.

The cases not covered by the Theorem are when $k \leq 3$. The case $k = 1$ was dealt with in [2]. When $k = 2$, we have $F_{m-2} + F_{m+2} = 3F_m$. Now $3 | F_m$ if and only if $4 | m$, from which we see that if $(3, F_m) = 1$ and $m > 3$ then

$$\phi(F_{m-2} + F_{m+2}) = \begin{cases} 4m \text{ if } m \text{ is even} \\ 8m \text{ if } m \text{ is odd.} \end{cases}$$

When $k = 3$, then $F_{m-3} + F_{m+3} = 2H_m$. Now $2|H_m$ if and only if $3|m$. Thus, if $(2,H_m) = 1$ we have that

$$\phi(F_{m+3} + F_{m-3}) = \begin{cases} 12m \text{ if } m \text{ is even} \\ 6m \text{ if } m \text{ is odd.} \end{cases}$$

Finally, it may be worthwhile commenting on the conditions of the form $(H_a, F_b) = 1$ which have been necessary for our computations. $(H_a, F_b) > 1$ is not a rare phenomenon because, for instance, given a it is easy to determine an infinite number of values of b for which $H_a|F_b$. In fact, as we now show, $H_a|F_b$ if and only if b is a positive integral multiple of $2a$. For, $H_a|F_{2a}$ because $F_{2a} = F_a H_a$. Thus, $H_a|F_{2ac}$ for any positive integer c. Actually, $2a$ is the least suffix b for which $H_a|F_b$, as shown by the proof of Theorem B in [2]. Let B denote the set of all positive integers b for which $H_a|F_b$. Then B is nonempty, and if $b_1, b_2 \varepsilon B$ since

$$F_{b_1+b_2} = F_{b_1+1}F_{b_2} + F_{b_1}F_{b_2-1}$$
$$F_{b_1-b_2} = (-1)^{b_2}(F_{b_2-1}F_{b_1} - F_{b_2}F_{b_1-1}),$$

we see that $b_1 + b_2,\ b_1 - b_2 \varepsilon B$. This means that B consists of all multiples of some least element which, as already pointed out, is $2a$ (see Theorem 6 in Chapter I of [4]).

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MUTUALLY COUNTING SEQUENCES

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ABSTRACT

Let n and m be positive integers with $n \le m$. Let A be the sequence of n nonnegative integers $a(0), a(1), \ldots, a(n-1)$, and let B be the sequence of m nonnegative integers $b(0), b(1), \ldots, b(m-1)$, where $a(i)$ is the multiplicity of i in B and $b(j)$ is the multiplicity of j in A. We prove that for $n > 7$, there are exactly 3 ways to generate such pairs of sequences.

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Let n and m be positive integers with $n \le m$. Let A be the sequence of n nonnegative integers $a(0), a(1), \ldots, a(n-1)$, and let B be the sequence of m nonnegative integers $b(0), b(1), \ldots, b(m-1)$, where $a(i)$ is the multiplicity of i in B and $b(j)$ is the multiplicity of j in A. Then A and B