PERFECT MAGIC CUBES OF ORDER $4m$

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ABSTRACT

It has long been known that there exists a perfect magic cube of order n where $n \neq 3, 5, 7, 2m$, and $4m$ with m odd and $m \geq 7$. That they do not exist for orders 2, 3, and 4 is not difficult to show. Recently, several authors have constructed perfect magic cubes of order 7. We shall give a method for constructing perfect magic cubes of orders $n = 4m$ with m odd and $m \geq 7$.

1. INTRODUCTION

A magic square of order n is an $n \times n$ arrangement of the integers $1, 2, \ldots, n^2$ so that the sum of the integers in every row, column and the two main diagonals is $n(n^2+1)/2$: the magic sum. Magic squares of orders 5 and 6 are shown in Figure 1.

20	22	4	6	13
9	11	18	25	2
23	5	7	14	16
12	19	21	3	10
1	8	15	17	24

1	34	33	32	9	2
29	11	18	20	25	8
30	22	23	13	16	7
6	17	12	26	19	31
10	24	21	15	14	27
35	3	4	5	28	36

Fig. 1

It is a well-known and long established fact that there exists a magic square of every order n, $n \neq 2$. For details of these constructions, the reader is referred to W. S. Andrews [2], Maurice Kraitchik [13], and W. W. Rouse Ball [6].

We can extend the concept of magic squares into three dimensions. A *magic cube* of order n is an $n \times n \times n$ arrangement of the integers $1, 2, \ldots, n^3$ so that the sum of the integers in every row, column, file and space diagonal (of which there are four) is $n(n^3+1)/2$: the magic sum. A magic cube of order 3 is exhibited in Figure 2.

10	26	6
24	1	17
8	15	19

23	3	16
7	14	21
12	25	5

9	13	20
11	27	4
22	2	18

Fig. 2

Magic cubes can be constructed for every order n, $n \neq 2$ (see W. S. Andrews [2]). A *perfect magic cube* of order n is a magic cube of order n with the additional property that the sum of the integers in the main diagonals of every layer parallel to a face of the cube is also $n(n^3+1)/2$. In 1939 Barkley Rosser and R. J. Walker [15] showed that there exists a perfect magic cube of order n, $n \neq 3$, 5, 7, $2m$, or $4m$, m odd. In fact, they constructed diabolic magic cubes of order n and showed that they exist only when $n \neq 3$, 5, 7, $2m$, or $4m$, m odd. A *diabolic* (or *pandiagonal*) *magic cube* of order n is a magic cube of order n in which the sum of the integers in every diagonal, both broken and unbroken, is $n(n^3+1)/2$. Clearly, a diabolic magic cube is also a perfect magic cube. We shall prove that there do not exist perfect magic cubes of orders 3 and 4. These proofs are due to Lewis Myers, Jr. [9] and Richard Schroeppel [9]. Perfect magic cubes of order 7 are known to have been constructed by Ian P. Howard, Richard Schroeppel, Ernst G. Straus, and Bayard E. Wynne. In this paper, we shall present a construction for perfect magic cubes of orders $n = 4m$, m odd, $m \geq 7$, leaving only the orders $n = 5$, 12, 20, and $2m$ for m odd to be resolved. We remark here that Schroeppel has shown that if a perfect magic cube of order 5 exists, then its center must be 63.

2. DEFINITIONS and CONSTRUCTIONS

As far as possible, the definitions will be in accord with those given in J. Dénes and A. D. Keedwell [7].

An $n \times n \times n$ three-dimensional matrix comprising n files, each having n rows and n columns, is called a *cubic array* of order n. We shall write this array as $A = (a_{ijk})$, $i, j, k \in \{1, 2, \ldots, n\}$ where a_{ijk} is the element in the ith row, jth column, and kth file of the array. When we write $a_{i+r,\, j+s,\, k+t}$ we mean for the indices $i + r$, $j + s$, and $k + t$ to be calculated modulo n on the residues $1, 2, \ldots, n$.

The set of n elements $\{a_{i+\ell,\, j,\, k}\colon\ \ell = 1, 2, \ldots, n\}$ constitutes a column; $\{a_{i,\, j+\ell,\, k}\colon\ \ell = 1, 2, \ldots, n\}$ constitutes a row; $\{a_{i,\, j,\, k+\ell}\colon\ \ell = 1, 2, \ldots, n\}$ constitutes a file; and $\{a_{i+\ell,\, j+\ell,\, k}\colon\ \ell = 1, 2, \ldots, n\}$, $\{a_{i+\ell,\, j,\, k+\ell}\colon\ \ell = 1, 2, \ldots, n\}$, $\{a_{i,\, j+\ell,\, k+\ell}\colon\ \ell = 1, 2, \ldots, n\}$ and $\{a_{i+\ell,\, j+\ell,\, k+\ell}\colon\ \ell = 1, 2, \ldots, n\}$ constitute the diagonals. Note that a diagonal is either broken or unbroken; being unbroken if all n of its elements lie on a straight line. The unbroken diagonals consist of the main diagonals, of which there are two in every layer parallel to a face of the cube, and the four space diagonals.

We shall distinguish three types of layers in a cube. There are those with fixed row, fixed column, or fixed file. The first we shall call the CF-layers, the ith CF-layer consisting of the n^2 elements $\{a_{ijk}\colon\ 1 \leq j, k \leq n\}$. The secone are the RF-layers in which the jth RF-layer consists of the n^2 elements $\{a_{\ jk}\colon\ 1 \leq i, k \leq n\}$. And finally, the RC-layers, the kth consisting of the elements $\{a_{ijk}\colon\ 1 \leq i, j \leq n\}$.

A cubic array of order n is called a *Latin cube* of order n if it has n distinct elements each repeated n^2 times and so arranged that in each layer parallel to a face of the cube all n distinct elements appear, and each is repeated exactly n times in that layer. In the case when each layer parallel to a face of the cube is a Latin square, we have what is called a *permutation cube* of order n. From this point on, we shall be concerned only with Latin cubes (and permutation cubes) based on the integers $1, 2, \ldots, n$.

Three Latin cubes of order n, $A = (a_{ijk})$, $B = (b_{ijk})$, and $C = (c_{ijk})$, are said to be *orthogonal* if among the n^3 ordered 3-tuples of elements $(a_{ijk}, b_{ijk}, c_{ijk})$ every distinct ordered 3-tuple involving the integers $1, 2, \ldots, n$ occurs exactly once. Should A, B, and C be orthogonal permutation cubes, they are said to form a *variational cube*. We shall write $D = (d_{ijk})$ where $d_{ijk} = (a_{ijk}, b_{ijk}, c_{ijk})$ to be the cube obtained on superimposing the Latin cubes A, B, and C and will denote it by $D = (A, B, C)$.

A cubic array of order n in which each of the integers $1, 2, \ldots, n^3$ occurs exactly once and in which the sum of the integers in every row, column, file, and unbroken diagonal is $n(n^3 + 1)/2$ is called a *perfect magic cube*.

We shall give two methods for constructing perfect magic cubes. These methods form the basis on which the perfect magic cubes of order $n = 4m$, m odd, $m \geq 7$, of Section 3 will be constructed.

Construction 1: Let $A = (a_{ijk})$, $B = (b_{ijk})$, and $C = (c_{ijk})$ be three orthogonal Latin cubes of order n with the property that in each cube the sum of the integers in every row, column, file, and unbroken diagonal is $n(n + 1)/2$. Then the cube $E = (e_{ijk})$ where $e_{ijk} = n^2(a_{ijk} - 1) + n(b_{ijk} - 1) + (c_{ijk} - 1) + 1$ is a perfect magic cube of order n. This is verified by checking that each of the integers $1, 2, \ldots, n^3$ appears in E and that the sum of the integers in every row, column, file, and unbroken diagonal is $n(n^3 + 1)/2$. It is clear that each of $1, 2, \ldots, n^3$ appears exactly once in E. We shall show that the sum of the integers in any row of E is $n(n^3 + 1)/2$. The remaining sums can be checked in a similar manner.

$$\begin{aligned}
\sum_{\ell=1}^{n} e_{i,j+\ell,k} &= \sum_{\ell=1}^{n} (n^2 (a_{i,j+\ell,k} - 1) + n(b_{i,j+\ell,k} - 1) + (c_{i,j+\ell,k} - 1) + 1) \\
&= n^2 \sum_{\ell=1}^{n} a_{i,j+\ell,k} + n\sum_{\ell=1}^{n} b_{i,j+\ell,k} + \sum_{\ell=1}^{n} c_{i,j+\ell,k} - \sum_{\ell=1}^{n} (n^2 + n) \\
&= (n^2 + n + 1)(n(n+1)/2) - n(n^2 + n) \\
&= n(n^3 + 1)/2.
\end{aligned}$$

Construction 2: Let $A = (a_{ijk})$ and $B = (b_{ijk})$ be perfect magic cubes of orders m and n, respectively. Replace b_{ijk} in B by the cube $C = (c_{rst})$ where $c_{rst} = a_{rst} + m^3(b_{ijk} - 1)$. This results in a perfect magic cube $E = (e_{ijk})$ of order nm. Each of the integers

$$1, 2, \ldots, m^3, m^3 + 1, m^3 + 2, \ldots, 2m^3, \ldots,$$
$$(n^3 - 1)m^3 + 1, (n^3 - 1)m^3 + 2, \ldots, n^3m^3$$

appears exactly once in E. As in the first construction, we shall show that the row sum in E is $nm((nm)^3 + 1)/2$; the remaining sums are similarly verified.

$$\begin{aligned}
\sum_{\ell=1}^{nm} e_{u,v+\ell,w} &= n\sum_{\ell=1}^{m} a_{r,s+\ell,t} + m\sum_{\ell=1}^{n} m^3 (b_{i,j+\ell,k} - 1) \\
&= nm(m^3 + 1)/2 + m^4(n(n^3 + 1)/2 - n) \\
&= nm((nm)^3 + 1)/2.
\end{aligned}$$

It will be seen in Theorem 3.6 that it is not necessary that A and B should both be perfect magic cubes in order for E to be a perfect magic cube.

3. PERFECT MAGIC CUBES

The first result is stated without proof and is due to Barkley Rosser and R. J. Walker [15].

Theorem 3.1: There exists a perfect magic cube of order n provided $n \neq 3, 5, 7, 2m$, or $4m$ for m odd.□

The following three theorems are the only known nonexistence results for perfect magic cubes. For the first, the proof is trivial. The proof of the second theorem is that of Lewis Myers, Jr. (see [9]) and of the third is that of Richard Schroeppel (also see [9]).

Theorem 3.2: There is no perfect magic cube of order 2.□

Theorem 3.3: There is no perfect magic cube of order 3.

Proof: Let $A = (a_{ijk})$ be a perfect magic cube of order 3; the magic sum is 42. The following equations must all hold:

$$a_{11k} + a_{22k} + a_{33k} = a_{13k} + a_{22k} + a_{31k} = a_{12k} + a_{22k} + a_{32k} = 42$$
and $$a_{11k} + a_{12k} + a_{13k} = a_{31k} + a_{32k} + a_{33k} = 42.$$

But together these imply that $a_{22k} = 14$ for $k = 1, 2$, and 3, a contradiction.□

Theorem 3.4: There is no perfect magic cube of order 4.

Proof: Let $A = (a_{ijk})$ be a perfect magic cube of order 4; the magic sum is 130.

First, we shall show that in any layer of such a cube the sum of the four corner elements is 130. Consider the kth RC-layer. The following equations must hold in A:

$$\begin{aligned} a_{11k} + a_{12k} + a_{13k} + a_{14k} &= a_{11k} + a_{22k} + a_{33k} + a_{44k} \\ &= a_{11k} + a_{21k} + a_{31k} + a_{41k} = 130, \\ a_{14k} + a_{23k} + a_{32k} + a_{41k} &= a_{14k} + a_{24k} + a_{34k} + a_{44k} \\ &= a_{41k} + a_{42k} + a_{43k} + a_{44k} = 130. \end{aligned}$$

These imply that

$$2(a_{11k} + a_{14k} + a_{41k} + a_{44k}) + \sum_{i=1}^{4} \sum_{j=1}^{4} a_{ijk} = 6 \cdot 130$$

and as

$$\sum_{i=1}^{4} \sum_{j=1}^{4} a_{ijk} = 4 \cdot 130,$$

then

$$a_{11k} + a_{14k} + a_{41k} + a_{44k} = 130.$$

Since the same argument holds for any type of layer in the cube, we have that the sum of the four corner elements in any layer is 130. A similar argument shows that $a_{111} + a_{114} + a_{441} + a_{444} = 130$. Thus we have

$$\begin{aligned} a_{111} + a_{114} + a_{144} + a_{141} &= a_{141} + a_{144} + a_{444} + a_{441} \\ &= a_{111} + a_{114} + a_{444} + a_{441} = 130, \end{aligned}$$

from which it follows that

$$a_{111} + a_{114} + a_{144} + a_{141} + 2(a_{444} + a_{441}) = 260,$$

and hence $a_{444} + a_{441} = 65$. Similarly, we can show that $a_{141} + a_{441} = 65$. Combining these two results, we have $a_{141} = a_{444}$, a contradiction.□

Using an argument similar to that of Theorem 3.4 Schroeppel has shown that if there exists a perfect magic cube of order 5 its center is 63.

For some time it was not generally known whether or not there existed a perfect magic cube of order 7 but when, in 1976, Martin Gardner [9] asked for such a cube, it appeared that they had been constructed without difficulty by many authors including Schroeppel, Ian P. Howard, Ernst G. Straus, and Bayard E. Wynne [17].

Theorem 3.5: There exists a perfect magic cube of order 7.

Proof: We shall construct a variational cube of order 7 from which a perfect magic cube of order 7 can be obtained via Construction 1. Let the three cubes forming the variational cube be A, B, and C; the first RC-layer of each being shown in Figure 3. Complete A, B, and C using the defining relations

$$a_{i,j,k+1} = a_{ijk} + 1, \; b_{i,j,k+1} = b_{ijk} + 1$$

and

$$c_{i,j,k+1} = c_{ijk} + 2,$$

where the addition is modulo 7 on the residues 1, 2, ..., 7. Now, in A and B, exchange the integers 4 and 7 throughout each cube. The variational cube of order 7 now has the properties required by Construction 1 and so we can construct a perfect magic cube of order 7. This can easily be checked.□

1	4	7	3	6	2	5	1	6	4	2	7	5	3	1	3	5	7	2	4	6
6	2	5	1	4	7	3	4	2	7	5	3	1	6	7	2	4	6	1	3	5
4	7	3	6	2	5	1	7	5	3	1	6	4	2	6	1	3	5	7	2	4
2	5	1	4	7	3	6	3	1	6	4	2	7	5	5	7	2	4	6	1	3
7	3	6	2	5	1	4	6	4	2	7	5	3	1	4	6	1	3	5	7	2
5	1	4	7	3	6	2	2	7	5	3	1	6	4	3	5	7	2	4	6	1
3	6	2	5	1	4	7	5	3	1	6	4	2	7	2	4	6	1	3	5	7

Fig. 3

We shall now proceed to the main theorem.

Theorem 3.6: There exists a perfect magic cube of order $4m$ for m odd and $m \geq 7$.

Proof: We know that there exists a perfect magic cube of order m, m odd and $m \geq 7$. This follows from Theorems 3.1 and 3.5. Since there does not exist a perfect magic cube of order 4 (Theorem 3.4), we cannot simply appeal to Construction 2 and obtain the desired perfect magic cubes. However, we can use Construction 2 and by a suitable arrangement of cubes of order 4 obtain a perfect magic cube of order $4m$. The construction is as follows.

Let $A = (a_{ijk})$ be a perfect magic cube of order m, m odd and $m \geq 7$. Let $B = (b_{ijk})$ be a cubic array of order m in which each b_{ijk} is some cubic array D_{ijk} of order 4 whose entries are ordered 3-tuples from the integers 1, 2, 3, 4 with every such 3-tuple appearing exactly once. The D_{ijk} are to be chosen in such a way that in the cubic array B the componentwise sum of the integers in every row, column, file, and unbroken diagonal is $(10m, 10m, 10m)$. It is now a simple matter to produce a perfect magic cube of order 4 . In D_{ijk} replace the 3-tuple (r, s, t) by the integer

$$(16(r - 1) + 4(s - 1) + (t - 1) + 1) + 64(a_{ijk} - 1).$$

The cubic array $E = (e_{ijk})$ of order $4m$ so constructed is, by considering Constructions 1 and 2, a perfect magic cube.

It remains then to determine the order 4 cubic arrays D_{ijk}.

Consider the four Latin cubes X_1, X_2, X_3, and X_4 as shown in Figure 4 where from left to right we have the first to the fourth RC-layers. It is not difficult to check that X_1, X_2, and X_3 are orthogonal, as are X_1, X_2, and X_4. We shall write X_i^* for the Latin cube X_i in which the integers 1 and 4 have been exchanged as have 2 and 3. Also $(X_1, X_2, X_3)'$ means that the cubic array (X_1, X_2, X_3) has been rotated forward through 90° so that RC-layers have become CF-layers, CF-layers have become RC-layers, and the roles of rows and files have interchanged in RF-layers.

X_1:	1	1	4	4	3	3	2	2	4	4	1	1	2	2	3	3
	4	4	1	1	2	2	3	3	1	1	4	4	3	3	2	2
	2	2	3	3	4	4	1	1	3	3	2	2	1	1	4	4
	3	3	2	2	1	1	4	4	2	2	3	3	4	4	1	1
X_2:	1	4	2	3	4	1	3	2	2	3	1	4	3	2	4	1
	1	4	2	3	4	1	3	2	2	3	1	4	3	2	4	1
	4	1	3	2	1	4	2	3	3	2	4	1	2	3	1	4
	4	1	3	2	1	4	2	3	3	2	4	1	2	3	1	4
X_3:	4	4	1	1	2	2	3	3	3	3	2	2	1	1	4	4
	1	1	4	4	3	3	2	2	2	2	3	3	4	4	1	1
	2	2	3	3	4	4	1	1	1	1	4	4	3	3	2	2
	3	3	2	2	1	1	4	4	4	4	1	1	2	2	3	3

(continued)

X_4:	3 3 2 2				1 1 4 4				4 4 1 1				2 2 3 3			
	2 2 3 3				4 4 1 1				1 1 4 4				3 3 2 2			
	1 1 4 4				3 3 2 2				2 2 3 3				4 4 1 1			
	4 4 1 1				2 2 3 3				3 3 2 2				1 1 4 4			

Fig. 4

We can now define the cubic arrays D_{ijk}.

$$D_{1,\frac{m+1}{2},\frac{m+1}{2}} = D_{m,\frac{m+1}{2},\frac{m+1}{2}} = D_{i,1,m} = D_{i,m,m} = (X_1, X_2, X_3)', \quad i = 2, 3, \ldots, m-1$$

$$D_{1,1,m} = D_{1,m,m} = D_{1,1,1} = D_{1,m,1} = (X_1, X_2, X_3)$$

$$D_{2,2,m-1} = D_{2,m-1,m-1} = (X_2^*, X_3, X_1^*)$$

$$D_{2,2,2} = D_{2,m-1,2} = (X_2, X_4, X_1)$$

$$D_{3,3,m-2} = D_{3,m-2,m-2} = (X_3^*, X_1^*, X_2^*)$$

$$D_{3,3,3} = D_{3,m-2,3} = (X_3, X_1, X_2)$$

$$D_{i,m+1-i,i} = D_{i,m+1-i,m+1-i} = D_{i,i,m+1-i} = D_{i,i,i} = (X_1, X_2, X_3), \quad i = 4, 5, \ldots, \frac{m+3}{2}$$

$$D_{i,m+1-i,i} = D_{i,m+1-i,m+1-i} = D_{i,i,m+1-i} = D_{i,i,i} = (X_1^*, X_2^*, X_3^*), \quad i = \frac{m+5}{2}, \frac{m+7}{2}, \ldots, m.$$

In every CF-layer of B, except for the second and third, replace the remaining b_{ijk} in each unbroken diagonal by either

$$D_{ijk} = (X_1, X_2, X_3) \text{ or } D_{ijk} = (X_1^*, X_2^*, X_3^*)$$

so that in each diagonal there are $(m-1)/2$ arrays (X_1, X_2, X_3) and $(m-1)/2$ arrays (X_1^*, X_2^*, X_3^*). In the second and third layers do the same but here there are to be only $(m-3)/2$ of each type of array as already three arrays in each diagonal are determined. All remaining b_{ijk} are to be replaced by

$$D_{ijk} = (X_1, X_2, X_3).$$

We must now verify that in this cubic array the componentwise sum of the integers in every row, column, file, and unbroken diagonal is $(10m, 10m, 10m)$.

Since the sum of the integers in every row, column, and file of X_i and X_i^*, $i = 1, 2, 3, 4$, is 10, then in B the componentwise sum of the integers in every row, column, and file is $(10m, 10m, 10m)$. Also, as the sum of the integers in every unbroken diagonal in the RC-layers and RF-layers of X_i and X_i^*, $i = 1, 2, 3, 4$, is 10, and as $(X_1, X_2, X_3)'$ does not occur on any of these unbroken diagonals in B, then the componentwise sum of the integers in these unbroken diagonals of B is $(10m, 10m, 10m)$. So we now have only to check the sums on the unbroken diagonals of the CF-layers and the sums on the four space diagonals of B.

The unbroken diagonals in the CF-layers of B are $D_{i11}, D_{i22}, \ldots, D_{imm}$ and $D_{im1}, D_{i,m-1,2}, \ldots, D_{i1m}$, $i = 1, 2, \ldots, m$. Let us write $S_r(D_{ijk})$ for the componentwise sum of the integers in the relevant diagonal in the rth CF-layer of D_{ijk}. We want to show that

$$\sum_{j=1}^{m} S_r(D_{ijj}) = \sum_{j=1}^{m} S_r(D_{i,m+1-j,j}) = (10m, 10m, 10m), \quad r = 1, 2, 3, 4.$$

If $i \neq 2, 3$, then

$$\sum_{j=1}^{m} S_r(D_{ijj}) = \frac{m-1}{2} S_r((X_1, X_2, X_3)) + \frac{m-1}{2} S_r((X_1^*, X_2^*, X_3^*)) + S_r((X_1, X_2, X_3)')$$

$$= \begin{cases} \frac{m-1}{2}(8, 4, 12) + \frac{m-1}{2}(12, 16, 8) + (10, 10, 10) & \text{when } r = 1 \\ \frac{m-1}{2}(12, 4, 8) + \frac{m-1}{2}(8, 16, 12) + (10, 10, 10) & \text{when } r = 2 \\ \frac{m-1}{2}(12, 16, 12) + \frac{m-1}{2}(8, 4, 8) + (10, 10, 10) & \text{when } r = 3 \\ \frac{m-1}{2}(8, 16, 8) + \frac{m-1}{2}(12, 4, 12) + (10, 10, 10) & \text{when } r = 4 \end{cases}$$

$$= (10m, 10m, 10m).$$

Also,

$$\sum_{j=1}^{m} S_r(D_{2jj}) = \frac{m-3}{2} S_r((X_1, X_2, X_3)) + \frac{m-3}{2} S_r((X_1^*, X_2^*, X_3^*)) + S_r((X_2, X_4, X_1)) + S_r((X_2^*, X_3, X_1^*)) + S_r((X_1, X_2, X_3)')$$

$$= \begin{cases} \frac{m-3}{2}(8, 4, 12) + \frac{m-3}{2}(12, 16, 8) + (4, 8, 8) + (16, 12, 12) + (10, 10, 10) & \text{when } r = 1 \\ \frac{m-3}{2}(12, 4, 8) + \frac{m-3}{2}(8, 16, 12) + (4, 12, 12) + (16, 8, 8) + (10, 10, 10) & \text{when } r = 2 \\ \frac{m-3}{2}(12, 16, 12) + \frac{m-3}{2}(8, 4, 8) + (16, 8, 12) + (4, 12, 8) + (10, 10, 10) & \text{when } r = 3 \\ \frac{m-3}{2}(8, 16, 8) + \frac{m-3}{2}(12, 4, 12) + (16, 12, 8) + (4, 8, 12) + (10, 10, 10) & \text{when } r = 4 \end{cases}$$

$$= (10m, 10m, 10m)$$

and

$$\sum_{j=1}^{m} S_r(D_{3jj}) = \frac{m-3}{2} S_r((X_1, X_2, X_3)) + \frac{m-3}{2} S_r((X_1^*, X_2^*, X_3^*)) + S_r((X_3, X_1, X_2)) + S_r((X_3^*, X_1^*, X_2^*)) + S_r((X_1, X_2, X_3)')$$

$$= \begin{cases} \frac{m-3}{2}(8, 4, 12) + \frac{m-3}{2}(12, 16, 8) + (12, 8, 4) + (8, 12, 16) + (10, 10, 10) & \text{when } r = 1 \\ \frac{m-3}{2}(12, 4, 8) + \frac{m-3}{2}(8, 16, 12) + (8, 12, 4) + (12, 8, 16) + (10, 10, 10) & \text{when } r = 2 \\ \frac{m-3}{2}(12, 16, 12) + \frac{m-3}{2}(8, 4, 8) + (12, 12, 16) + (8, 8, 4) + (10, 10, 10) & \text{when } r = 3 \\ \frac{m-3}{2}(8, 16, 8) + \frac{m-3}{2}(12, 4, 12) + (8, 8, 16) + (12, 12, 4) + (10, 10, 10) & \text{when } r = 4 \end{cases}$$

$$= (10m, 10m, 10m).$$

Similarly, one can check that

$$\sum_{j=1}^{m} S_r(D_{i,m+1-j,j}) = (10m,\ 10m,\ 10m)$$

and so the componentwise sum of the integers in the unbroken diagonals in the CF-layers of B is $(10m, 10m, 10m)$.

The four space diagonals of B are

$$D_{iii},\ i = 1, 2, \ldots, m;\ D_{i,m+1-i,i},\ i = 1, 2, \ldots, m;$$
$$D_{m+1-i,i,i},\ i = 1, 2, \ldots, m;\ D_{m+1-i,m+1-i,i},\ i = 1, 2, \ldots, m.$$

Write $S(D_{ijk})$ to be the sum of the integers in the relevant space diagonal of D_{ijk}. We want to show that

$$\sum_{i=1}^{m} S(D_{iii}) = \sum_{i=1}^{m} S(D_{i,m+1-i,i}) = \sum_{i=1}^{m} S(D_{i,m+1-i,m+1-i}) = \sum_{i=1}^{m} S(D_{i,i,m+1-i})$$
$$= (10m,\ 10m,\ 10m).$$

Consider each of the space diagonals in turn.

$$\begin{aligned}\sum_{i=1}^{m} S(D_{iii}) &= S((X_1, X_2, X_3)) + S((X_2, X_4, X_1)) + S((X_3, X_1, X_2)) \\ &\qquad + \frac{m-3}{2}S((X_1, X_2, X_3)) + \frac{m-3}{2}S((X_1^*, X_2^*, X_3^*)) \\ &= (6, 10, 14) + (10, 14, 6) + (14, 6, 10) + \frac{m-3}{2}(6, 10, 14) \\ &\qquad + \frac{m-3}{2}(14, 10, 6) \\ &= (10m,\ 10m,\ 10m)\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{m} S(D_{i,m+1-i,i}) &= S((X_1, X_2, X_3)) + S((X_2, X_4, X_1)) + S((X_3, X_1, X_2)) \\ &\qquad + \frac{m-3}{2}S((X_1, X_2, X_3)) + \frac{m-3}{2}S((X_1^*, X_2^*, X_3^*)) \\ &= (14, 10, 6) + (10, 6, 14) + (6, 14, 10) + \frac{m-3}{2}(14, 10, 6) \\ &\qquad + \frac{m-3}{2}(6, 10, 14) \\ &= (10m,\ 10m,\ 10m)\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{m} S(D_{i,m+1-i,m+1-i}) &= S((X_1, X_2, X_3)) + S((X_2^*, X_3, X_1^*)) + S((X_3^*, X_1^*, X_2^*)) \\ &\qquad + \frac{m-3}{2}S((X_1, X_2, X_3)) + \frac{m-3}{2}S((X_1^*, X_2^*, X_3^*)) \\ &= (14, 10, 14) + (10, 14, 6) + (6, 6, 10) + \frac{m-3}{2}(14, 10, 14) \\ &\qquad + \frac{m-3}{2}(6, 10, 6) \\ &= (10m,\ 10m,\ 10m)\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{m} S(D_{i,i,m+1-i}) &= S((X_1, X_2, X_3)) + S((X_2^*, X_3, X_1^*)) + S((X_3^*, X_1^*, X_2^*)) \\ &\quad + \frac{m-3}{2} S((X_1, X_2, X_3)) + \frac{m-3}{2} S((X_1^*, X_2^*, X_3^*)) \\ &= (6, 10, 6) + (10, 6, 14) + (14, 14, 10) + \frac{m-3}{2}(6, 10, 6) \\ &\quad + \frac{m-3}{2}(14, 10, 14) \\ &= (10m, 10m, 10m).\end{aligned}$$

Thus we have found a way of arranging order 4 cubic arrays, in which each of the ordered 3-tuples on 1, 2, 3, 4 appears exactly once, in an order m cubic array B so that in B the componentwise sum of the integers in every row, column, file, and unbroken diagonal is $(10m, 10m, 10m)$. Therefore, as previously stated, we can construct a perfect magic cube of order $4m$ for m odd and $m \geq 7$. □

4. EXTENSIONS AND PROBLEMS

We know now that there exists a perfect magic cube of order n provided $n \neq$ 3, 4, 5, 12, 20, $2m$, for m odd, and that they do not exist when n = 2, 3, or 4. So the question remaining is whether or not there exist perfect magic cubes of orders n = 5, 12, 20, and $2m$, for m odd and $m \geq 3$. It seems probable that such cubes of orders 12 and 20 can be constructed along the lines of Theorem 3.6 using cubic arrays of orders 3 and 5 that are *close* to being perfect magic cubes and arranging in them order 4 cubic arrays composed from X_i and X_i^*, i = 1, 2, 3, 4, as before. It may also be possible that by arranging order 2 cubic arrays in order m cubic arrays, m odd and $m \geq 7$, one can obtain perfect magic cubes of order $2m$. As for order 5, all we know is that if there is a perfect magic cube of order 5 its center is 63.

A more recent problem in the study of magic cubes is that of extending them into k dimensions. For details on this problem and the related problem of constructing variational cubes in k dimensions, the reader is referred to [1], [3], [4], [5], [7], [8], [12], and [16].

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GENERATING FUNCTIONS FOR RECURRENCE RELATIONS

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1. INTRODUCTION

In a previous paper [3] the author gave explicit solutions for four recurrence relations. The first was a basic relation with special initial conditions. The solution was shown to be related to the decompositions of the integer n relative to the first m positive integers. The second basic relation then restricted the first so that the solution was related to the decomposition of n relative to a subset of the first m positive integers. Then the initial conditions for both were extended to any arbitrary values.

In the next section we shall give the generating functions for all four of these cases, starting with the initial condition of highest index. We also note the form of the function for arbitrary indices for the initial conditions. Finally, we give a second function that generates all the initial conditions.

In Section 3 we give a simple example of the fourth kind of relation. We determine the first few terms of this relation and then compute its generating function. Then we consider relations given in [1] and [2] and determine their generating functions.

2. THE BASIC GENERATING FUNCTION

We shall consider a recurrence relation defined by

$$G_t = \sum_{s=1}^{m} r_s G_{t-s};\ G_{1-m},\ \ldots,\ G_0 \text{ arbitrary.}$$

For notation, we shall refer to its generating function as $R_m(G;x)$. The first term generated will be G_0. Later, we shall give a second function that will start with G_{1-m}.

Theorem 2.1: The generating function for the recurrence relation G_n is as follows:

$$R_m(G;\ x) = \left(G_0 + \sum_{n=1}^{m-1}\ \sum_{s=n+1}^{m} r_s G_{n-s} x^n\right)\left(1 - \sum_{s=1}^{m} r_s x^s\right)^{-1}.$$

To prove that this does generate G_n, we set this equal to $\sum_{n=0}^{\infty} G_n x^n$ and then