$$u = 1/[L:Q] = 1/(8 \cdot 3^{3N-1}) = 1/(8 \cdot 3^{3M-4}) > 0. \tag{31}$$

5. CONCLUDING REMARK

Further problems concerning the residues of n^n (mod p), where $1 \leq n \leq p-1$, are obtaining better upper and lower bounds for the number of distinct residues appearing among $\{n^n\}$ and determing estimates for the number of times that residues other than ±1 may occur.

REFERENCES

1. Raymond Ayoub. *An Introduction to the Analytic Theory of Numbers*. Mathematical Surveys No. 10. Providence, R.I.: American Mathematical Society, 1963. Pp. 327-342.
2. Bruce C. Berndt & S. Chowla. "Zero Sums of the Legendre Symbol." *Nordisk Matematisk Tidskrift* 22 (1974):5-8.
3. Z. I. Borevich & I. R. Shafarevich. *Number Theory*. New York: Academic Press, 1066. Pp. 346-347,
4. Roger Crocker. "On a New Problem in Number Theory." *Amer. Math. Monthly* 73 (1966):355-357.
5. Roger Crocker. "On Residues of n^n." *Amer. Math. Monthly* 76 (1969):1028-1929.

*****

A GENERALIZATION OF A PROBLEM OF STOLARSKY

MARY E. GBUR

Texas A&M University, College Station, TX 77843

For fixed positive integer $k \geq 1$, we set

$$a_k = [\overline{k}] = \frac{k + \sqrt{k^2 + 4}}{2},$$

a real number with completely periodic continued fraction expansion and period of length one. For all integers $n \geq 1$, we use $f_k(n)$ to denote the nearest integer to na_k.

Using this notation, we define an array $(b_{i,j}^{(k)})$ as follows. The first row has

$$b_{1,1}^{(k)} = 1 \quad \text{and} \quad b_{1,j}^{(k)} = f\ (b_{1,j-1}^{(k)}), \text{ for all } j \geq 2.$$

After inductively setting $b_{i,1}^{(k)}$ to be the smallest integer that has not occurred in a previous row, we define the remainder of the ith row by

$$b_{i,j}^{(k)} = f_k(b_{i,j-1}^{(k)}), \text{ for all } j \geq 2.$$

K. Stolarsky [4] developed this array for $k = 1$, showed that each positive integer occurs exactly once in the array, and proved that any three consecutive entries of each row satisfy the Fibonacci recursion. The latter result can be viewed as a generalization of a result of V. E. Hoggatt, Jr. [3, Theorem III]. In Theorem 1, we prove an analogous result for general k.

Theorem 1: Each positive integer occurs exactly once in the array $(b_{i,j}^{(k)})$. Moreover, the rows of the array satisfy

$$b_{i,j+2}^{(k)} = kb_{i,j+1}^{(k)} + b_{i,j}^{(k)}, \text{ for all } i,\ j \geq 1.$$

Proof: By construction, each positive integer occurs at least once. For $m \neq n$ we have $|(n-m)a_k| > 1$ and so $f_k(m) \neq f_k(n)$. Since the first column entry is the smallest in any row, every positive integer occurs exactly once.

Since
$$b_{i,j+2}^{(k)} = f_k(b_{i,j+1}^{(k)}) = f_k(f_k(b_{i,j}^{(k)})),$$
it suffices to show that $f_k(f_k(m)) = kf_k(m) + m$, for all $m \geq 1$. For any $m \geq 1$, $f_k(m) = ma_k + r$ for some $|r| \leq 1/2$. Hence
$$(a_k - k)f_k(m) = \frac{1}{a_k}(ma_k + r) = m + \frac{r}{a_k}, \text{ where } \left|\frac{r}{a_k}\right| < \frac{1}{2a_k} < \frac{1}{2}.$$
This implies that $f_k(f_k(m)) = kf_k(m) + m$, completing the proof.

For all integers $i \geq 1$, we set
$$b_{i,0}^{(k)} = b_{i,2}^{(k)} - kb_{i,1}^{(k)}.$$
For $k = 1$, Stolarsky [4] considered the sequence $\{b_{i,0}^{(1)}\}$ of differences and asked whether or not the sequence was a subset of the union of the first and second columns of $(b_{i,j}^{(1)})$. The following theorem shows that, for $k \geq 2$, no analogous result can hold.

Theorem 2: For $k \geq 2$, every positive integer occurs at least $k - 1$ times in the sequence of differences. For $k = 1$, D is a difference if and only if D occurs twice in the sequence $\{f_1(n) - n\}$.

Proof: Fix $k \geq 1$. If we append the difference $b_{i,0}^{(k)}$ to the beginning of the ith row of the array, this new augmented array contains (with the same multiplicity) all the elements of the sequence with general term $f_k(n) - kn$; that is, the nearest integer to $n(a_k - k)$.

Since $a_k - k < 1/k$, every positive integer must occur at least k times in the sequence $\{f_k(n) - kn\}$. On the other hand, in Theorem 1 we showed that the unaugmented array contains each positive integer exactly once. Therefore, for $k \geq 2$, every positive integer occurs at least $k - 1$ times in $\{b_{i,0}^{(k)}\}$.

From $a_1 - 1 > 1/2$, we know that any integer can occur at most twice in the sequence $\{f_1(n) - n\}$. Then, since $\{b_{i,0}^{(1)}\}$ is the modification of $\{f_1(n) - n\}$ obtained by deleting one copy of the set of positive integers, D is a difference for $k = 1$ if and only if D occurs twice in $\{f_1(n) - n\}$.

J. C. Butcher [1] and M. D. Hendy [2] have independently proved Stolarsky's conjecture and also have shown that the sequence of differences, for $k = 1$, is exactly the union of the first and second columns. In the remainder of this note we give another proof of these facts.

The following is an extension of Lemma 1 in [1].

Lemma 1: For any positive integers i, k, $j + 1$,
$$\frac{1}{2a_k^j} < \left|b_{i,j}^{(k)}a_k - b_{i,j+1}^{(k)}\right| < \frac{1}{2a_k^{j-1}}.$$

Proof: By definition of $b_{i,1}^{(k)}$, the left inequality holds for $j = 0$. Also, since $b_{i,2}^{(k)}$ is the nearest integer of $b_{i,1}^{(k)}a_k$, we have
$$\left|b_{i,0}^{(k)}a_k - b_{i,1}^{(k)}\right| = a_k\left|b_{i,0}^{(k)} - b_{i,1}^{(k)}\frac{1}{a_k}\right| = a_k\left|b_{i,2}^{(k)} - b_{i,1}^{(k)}a_k\right| < \frac{a_k}{2},$$
proving the right inequality for $j = 0$.

From the recursion formula to Theorem 1 we obtain
$$b_{i,j}^{(k)}a_k - b_{i,j+1}^{(k)} = a_k(b_{i,j+2}^{(k)} - a_k b_{i,j+1}^{(k)}).$$
Therefore, the lemma follows by induction from our verification for $j = 0$.

We henceforth only consider $k = 1$. Therefore, we suppress the index k and let $a = a_1 = \dfrac{1 + \sqrt{5}}{2}$ for the remainder of this paper.

Definition: Let D be a positive integer. We call D *early* if either $D = b_{i,1}$ or $D = b_{i,2}$ for some $i \geq 1$. D is called *late* if $D = b_{i,j}$ for some $i \geq 1$, $j \geq 3$.

The following corollary is a consequence of Lemma 1 and Theorem 1.

Corollary: Let D be a positive integer. Then D is early if and only if

$$\min_n |D - na| > \frac{1}{2a};$$

D is late if and only if

$$\min_n |D - na| < \frac{1}{2a},$$

where each minimum is taken over the set of integers.

Proof: By Theorem 1 there exist i, $j \geq 1$ for which $D = b_{i,j}$. By definition, $b_{i,j+1}$ is the nearest integer to Da. Therefore, from Lemma 1,

$$\frac{1}{2a^j} < |b_{i,j}a - b_{i,j+1}| = \min_n |Da - n| < \frac{1}{2a^{j-1}}. \tag{1}$$

Since D is an integer,

$$\min_n |Da - n| = \min_n |D(a - 1) - n| = \frac{1}{a} \min_n |D - na|.$$

Hence, (1) implies that

$$\frac{1}{2a^{j-1}} < \min_n |D - na| < \frac{1}{2a^{j-2}},$$

completing the proof.

Lemma 2: For any three (two) consecutive integers, at most two (at least one) are differences. Also, if both $N \pm 1$ are not differences, then both $N - 2$ and $N - 3$ are differences.

Proof: First, we suppose that the three consecutive integers N, $N \pm 1$ are differences. Then by Theorem 2 there exists an integer b for which

$$\frac{b}{a} + \frac{1}{2} > N - 1 \quad \text{and} \quad \frac{b+5}{a} + \frac{1}{2} < N + 2;$$

that is,

$$\frac{b+5}{a} - \frac{3}{2} < N < \frac{b}{a} + \frac{3}{2},$$

which contradicts $a < 5/3$. Since $a > 3/2$, the alternative statement follows similarly from Theorem 3.

By the alternative statements, if neither $N \pm 1$ is a difference, then both N and $N - 2$ are differences. Therefore, if $N - 3$ were to occur only once we would contradict $a > 8/5$.

Theorem 3: D is a difference if and only if D is early.

Proof: We suppose, on the contrary, that there exists a smallest integer D for which the theorem fails.

First, we assume that D is early but not a difference. By Lemma 2, $D - 1$ must be a difference and, by our assumption on D, is therefore early. Let na be the smallest multiple of a greater than $D - 1$. Since both $D - 1$ and D are early and $a = 1 + 1/a$, the corollary implies that

$$D - 1 + \frac{1}{2a} < na < D - \frac{1}{2a}. \tag{2}$$

Hence

$$|(D - 2) - (n - 1)a| < \frac{1}{2a},$$

and so, by the corollary, $D - 2$ is late. From our assumption on D, we thus obtain that $D - 2$ is not a difference. Because neither D nor $D - 2$ is a difference, by Lemma 2 both $D - 3$ and $D - 4$ are differences; thus $D - 3$ and $D - 4$ are early, and (2) implies that

$$D - 4 + \frac{1}{2a} < (n - 2)a < D - 3 - \frac{1}{2a}.$$

Combining this with (2) and successively manipulating, we obtain

$$D + \frac{5}{2}a - \frac{9}{2} = D - 4 + 2a + \frac{1}{2a} < na < D - \frac{1}{2a} = D - \frac{a}{2} + \frac{1}{2},$$

$$na + \frac{a}{2} - \frac{1}{2} < D < na - \frac{5}{2}a + \frac{9}{2},$$

$$n + 1 - \frac{a}{2} = n + \frac{1}{2} - \frac{1}{2a} < \frac{D}{a} < n - \frac{5}{2} + \frac{9}{2a} = n + \frac{9}{2}a - 7. \tag{3}$$

On the other hand, by Theorem 2 there exists an integer b for which

$$\frac{b}{a} + \frac{1}{2} < D - 2 \quad\text{and}\quad \frac{b + 5}{a} + \frac{1}{2} > D + 1;$$

that is,

$$b - D + \frac{5}{2}a < Da - D = \frac{D}{a} < b - D + 5 - \frac{a}{2}.$$

Comparing this with (3), we have

$$n + 1 - \frac{a}{2} < b - D + 5 - \frac{a}{2} \quad\text{and}\quad n + \frac{9}{2}a - 7 > b - D + \frac{5}{2}a;$$

that is, the integer $b - d$ satisfies

$$n - 4 < b - D < n - 7 + 2a < n - 3,$$

a contradiction.

Therefore, D is a late difference. By the corollary there exists an integer n for which

$$|na - D| < \frac{1}{2a}.$$

Hence

$$|(n - 1)a - (D - 1)| > \frac{1}{2a}$$

and, by the corollary, $D - 1$ is early and so is a difference. Since $D - 1$ and D are both differences, Lemma 2 implies that $D - 2$ cannot be a difference and so is late. Therefore,

$$|(n - 1)a - (D - 2)| < \frac{1}{2a}.$$

Combining this with the previous inequality

$$|na - D| < \frac{1}{2a},$$

we have

$$D - \frac{1}{2a} < na < D - 2 + \frac{1}{2a} + a.$$

We manipulate this to get the following two sets of inequalities:

$$n - 4 + \frac{5}{2}a < \frac{D}{a} < n + 1 - \frac{a}{2}, \tag{4}$$

and

$$D - 4 + \frac{1}{2a} < D - \frac{1}{2a} - 2a < (n - 2)a < D - 2 + \frac{1}{2a} - a = D - 3 - \frac{1}{2a}.$$

The latter implies that both $D - 3$ and $D - 4$ are early and so, by assumption, are differences. Therefore, by Theorem 2 there exists an integer b for which

$$\frac{b}{a} + \frac{1}{2} > D - 4 \quad \text{and} \quad \frac{b + 8}{a} + \frac{1}{2} < D + 1;$$

that is,

$$b - D + 8 - \frac{a}{2} < \frac{D}{a} < b - D + \frac{9}{2}a.$$

Comparing this with (4), we obtain

$$n - 8 < n - 4 - 2a < b - D < n - 7,$$

which is contrary to the fact that $b - D$ and n are integers.

ACKNOWLEDGMENT

The author would like to thank Gerald Bergum for his aid in increasing the readability of this paper.

REFERENCES

1. J. C. Butcher. "On a Conjecture Concerning a Set of Sequences Satisfying The Fibonacci Difference Equation." *The Fibonacci Quarterly* 16 (1978):81-83.
2. M. D. Hendy. "Stolarsky's Distribution of Positive Integers." *The Fibonacci Quarterly* 16 (1978):70-80.
3. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton Mifflin, 1969. Pp. 34-35.
4. K. Stolarsky. "A Set of Generalized Fibonacci Sequences Such That Each Natural Number Belongs to Exactly One." *The Fibonacci Quarterly* 15 (1977): 224.

*****

INITIAL DIGITS IN NUMBER THEORY

J. KNOPFMACHER

University of the Witwatersrand, Johannesburg, 2001, South Africa

INTRODUCTION

It has been observed empirically by various authors (cf. Raimi [5] and his references) that the numbers in "random" tables of physical or other data tend to begin with low digits more frequently than one might on first consideration expect. In fact, in place of the plausible-looking frequency of 1/9, it is found that for the numbers with first significant digit equal to

$$a \in \{1, 2, \ldots, 9\}$$

in any particular table the observed proportion is often approximately equal to

$$\log_{10}\left(1 + \frac{1}{a}\right).$$

A variety of explanations have been put forward for this surprising phenomenon.

Although more general cases have also been considered, most people might agree that it should suffice to consider only sets of positive integers, since empirical data are normally listed in terms of finite lists of numbers with finite decimal expansions (for which the signs or positions of decimal points are immaterial here). On accepting this simplification, the common tendency