inaccurate to discuss the fraction of F with first digit p. However, what we proved was that lim inf $\frac{1}{N}A_p(N) \geq \log\left(\frac{p+1}{p}\right)$. By the remark at the end of the proof, it is then easy to see that it is impossible to have lim sup $\frac{1}{N}A_p(N)$ greater than $\log\left(\frac{p+1}{p}\right)$ for any p. Therefore, lim sup = lim inf and the limit exists.

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A SIMPLE DERIVATION OF A FORMULA FOR $\sum_{k=1}^{n} k^r$

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The formula for

$$\sum_{k=1}^{n} k^r \quad (r \text{ and } n \text{ being positive integers})$$

is known (see Barnard & Child [1] and Jordan [2]). However, most undergraduate texts in algebra and calculus give these formulas only for $r = 1, 2,$ and 3. Perhaps the reason is that the known formula for general integral r is a bit involved and requires some background in the theory of polynomials and Bernoulli numbers. In this note we give a very simple derivation of this formula and no background beyond the knowledge of binomial theorem (integral power) and some elementary facts from calculus are needed. Consequently, the author hopes that the general formula can be exposed to undergraduates at some proper level.

Let

$$S_r(n) = \sum_{k=1}^{n} k^r,$$

where $r = 0, 1, \ldots, n = 1, 2, \ldots,$ and note that $S_0(n) = n$. In order to find a formula for $S_r(n)$, we use the following identity: For any integer k we have

$$\begin{aligned}\int_{k-1}^{k} x^r dx &= \frac{1}{r+1}(k^{r+1} - (k-1)^{r+1}) \\ &= \frac{1}{r+1}\sum_{j=0}^{r}\binom{r+1}{j}(-1)^{r+2-j}k^j \\ &= \sum_{j=0}^{r} a_j(r)k^j,\end{aligned}$$

where $a_j(r) = (-1)^{r+2-j}\binom{r+1}{j}\Big/(r+1)$. Hence,

$$\sum_{k=1} \int_{k-1}^{k} x^r dx = \int_0^n x^r dx = \sum_{j=0}^{r} a_j(r) S_j(n),$$

and

(1) $$\frac{n^{r+1}}{r+1} = \sum_{j=0}^{r} a_j(r) S_j(n).$$

Since $a_r(r) = 1$, it follows from (1) that

(2) $$S_r(n) = \frac{n^{r+1}}{r+1} - \sum_{j=0}^{r-1} a_j(r) S_j(n).$$

The numbers $a_j(r)$ can be easily evaluated. Here we list some of the $a_j(r)$'s:

$a_0(1) = -\frac{1}{2}$

$a_0(2) = \frac{1}{3},\ a_1(2) = -1$

$a_0(3) = -\frac{1}{4},\ a_1(3) = 1,\ a_2(3) = -\frac{3}{2}$

$a_0(4) = \frac{1}{5},\ a_1(4) = -1,\ a_2(4) = 2,\ a_3(4) = -2$

$a_0(5) = -\frac{1}{6},\ a_1(5) = 1,\ a_2(5) = -\frac{5}{2},\ a_3(5) = \frac{10}{3},\ a_4(6) = -\frac{5}{2}$

$a_0(6) = \frac{1}{7},\ a_1(6) = -1,\ a_2(6) = 3,\ a_3(6) = -5,\ a_4(6) = 5,\ a_5(6) = -3$, etc.

Using (2) we obtain

$$S_1(n) = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

$$S_2(n) = \frac{n^3}{3} - \left(\frac{n}{3} - \frac{n(n+1)}{2}\right) = \frac{n(n+1)(2n+1)}{6}$$

$$S_3(n) = \frac{n^4}{4} - \left(-\frac{n}{4} + \frac{n(n+1)}{2} - \frac{3}{2}\,\frac{n(n+1)(2n+1)}{6}\right) = \left(\frac{n(n+1)}{2}\right)^2.$$

Continuing in this fashion we obtain

$$S_4(n) = n(n+1)(2n+1)(3n^2+3n-1)/30$$
$$S_5(n) = n^2(n+1)^2(2n^2+2n-1)/12$$
$$S_6(n) = n(n+1)(2n+1)(3n^4+6n^3-3n-1)/42.$$

However, such evaluations get messy with higher values of r. An integral formula for $S_r(n)$ is known (cf. Barnard & Child [1]), but its evaluation depends on Bernoulli numbers. We derive this formula from (2) with an advantage that the required Bernoulli numbers satisfy a simple recurrence relation in terms of $a_j(r)$ which is a by-product of our derivation.

Treating n as a continuous variable and differentiating (2) with respect to n we have

(3) $$S_r'(n) = n^r - \sum_{j=0}^{r-1} a_j(r) S_j'(n),$$

where $S_j'(n) = \dfrac{dS_j(n)}{dn}$.

Since $S_j(n) - S_j(n-1) = n^j$, one obtains

$$S_j'(n) = j\,n^{j-1} + S_j'(n-1) = \cdots$$
$$= j\,n^{j-1} + j(n-1)^{j-1} + \cdots + j\,1^{j-1} + S_j'(0).$$

Clearly, $S_j'(0)$ is the coefficient of n in $S_j(n)$, and writing $B_j = S_j'(0)$, where $B_0 = 1$ and $S_0(n) = n$, we obtain

$$S_j'(n) = j\sum_{k=1}^{n} k^{j-1} + B_j = j\,S_{j-1}(n) + B_j. \tag{4}$$

From (3) and (4) we obtain

$$S_r'(n) = n^r - \sum_{j=0}^{r-1} j\,a_j(r)S_{j-1}(n) - \sum_{j=0}^{r-1} a_j(r)B_j.$$

It is easy to verify that

$$j\,a_j(r) = r\,a_{j-1}(r-1),$$

and hence

$$S_r'(n) = n^r - \sum_{j=1}^{r-1} r\,a_{j-1}(r-1)S_{j-1}(n) - \sum_{j=0}^{r-1} a_j(r)B_j$$
$$= r\left[\frac{n^r}{r} - \sum_{j=0}^{r-2} a_j(r-1)S_j(n)\right] - \sum_{j=0}^{r-1} a_j(r)B_j. \tag{5}$$

Thus it follows from (2) and (5) that

$$S_r'(n) = r\,S_{r-1}(n) + B_r, \tag{6}$$

where

$$B_r = -\sum_{j=0}^{r-1} a_j(r)B_j, \quad B_0 = 1. \tag{7}$$

The relation (6) immediately leads to

$$S_r(n) = r\int S_{r-1}(n)\,dn + n\,B_r. \tag{8}$$

The numbers B_r ($r = 0, 1, \ldots$) are Bernoulli numbers and can be generated from (7), and starting with $S_0(n) = n$ one obtains $S_r(n)$ from (8) for any desired r. Note that the relation (7) for Bernoulli numbers is a by-product of our derivation of (8) from (2). Consequently, no background in the theory of polynomials and Bernoulli numbers is needed to arrive at (8). Moreover, (7) and (8) together make it possible to evaluate $S_r(n)$ for any r, or one can use (8) to get an explicit expression for $S_r(n)$ (see Barnard & Child [1]).

To illustrate the preceding, from the list of $a_j(r)$ and (7) we easily obtain

$$B_0 = 1,\ B_1 = \frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42},\ \ldots,$$

and since $S_0(n) = n$, it follows from (8) that

$$S_1(n) = \int n\,dn + \frac{n}{2} = \frac{n^2}{2} + \frac{n}{2},$$
$$S_2(n) = 2\int\left(\frac{n^2}{2} + \frac{n}{2}\right)dn + \frac{n}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}, \text{ etc.}$$

Finally, we note the following interesting fact. Since

$$a_0(r) = \pm\frac{1}{r+1}$$

and

$$S_0(n) = n,$$

it follows from (2) that

$$S_r(n) = S_1(n)P_{r-1}(n),$$

where $P_{r-1}(n)$ is a polynomial in n of degree $r - 1$.

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A NOTE ON THE POLYGONAL NUMBERS

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1. INTRODUCTION

Polygonal numbers of order k ($k = 3, 4, 5, \ldots$) are the numbers

$$P_{n,k} = \frac{1}{2}[(k-2)n^2 - (k-4)n] \quad (n = 1, 2, 3, \ldots). \tag{1}$$

If $k = 4$, they are reduced to the square numbers. It is clear that there are an infinite number of square numbers which are at a time the sum and difference and the product of such numbers, from the identity

$$(4m^2 + 1)^2 = (4m)^2 + (4m^2 - 1)^2$$

$$= (8m^4 + 4m^2 + 1)^2 - (8m^4 + 4m^2)^2,$$

and since there are an infinite number of composite numbers of the form $4m^2 + 1$ (for example, if $m = 5j + 1$, $4m^2 + 1$ is divisible by 5).

Sierpinski [1] proved that there are an infinite number of triangular numbers ($k = 3$) which are at a time the sum and the difference and the product of such numbers.

For $k = 5$, Hansen [2] proved that there are an infinite number of $P_{n,5}$ that can be expressed as the sum and the difference of such numbers.

O'Donnell [3] proved a similar result for $k = 6$, and conjectured that there will be a similar result for the general case.

In this paper it will be shown that their method of proof is valid for the general case, proving the following theorem.

Theorem: Let a and b be given integers such that $a \neq 0$ and $a \equiv b \pmod 2$, and let

$$A_n = \frac{1}{2}(an^2 + bn) \quad (n = 1, 2, 3, \ldots). \tag{2}$$