4. $L_k = L_{k-1} + L_{k-2}$; $L_0 = 2$, $L_1 = 1$

$$(x + 2x^2)(1 - x - x^2)^{-1} = x + 3x^2 + 4x^3 + 7x^4 + \cdots$$

or

$$(2 - x)(1 - x - x^2)^{-1} = 2 + x + 3x^2 + 4x^3 + 7x^4 + \cdots$$

(This is the Lucas sequence.)

5. $U_k = rU_{k-1} + sU_{k-2}$; U_0, U_1 arbitrary

$$(U_1x + U_0sx^2)(1 - rx - sx^2)^{-1} = U_1x + (rU_1 + sU_0)x^2 + \cdots$$

or

$$(U_0 + (U_1 - U_0)x)(1 - rx - sx^2)^{-1} = U_0 + U_1x + (rU_1 + sU_0)x^2 + \cdots$$

6. $T_n = rT_{n-1} + sT_{n-2} - rsT_{n-3}$; T_0, T_1, T_2 arbitrary

$$(T_2x^2 + (sT_1 - rsT_0)x^3 - rsT_1x^4)(1 - rx - sx^2 + rsx^3)^{-1}$$
$$= T_2x^2 + (rT_2 + sT_1 - rsT_0)x^3 + \cdots$$

or

$$(T_0 + (T_1 - rT_0)x + (T_2 - rT_1 - sT_0)x^2)(1 - rx - sx^2 + r2x^3)^{-1}$$
$$= T_0 + T_1x + T_2x^2 + (rT_2 + sT_1 - r\ T_0)x^3 + \cdots$$

From the solutions given in [2] and [1], it can be verified that we obtain the terms generated above.

The generating function given in Section 2 can be used to generate terms of any given recurrence relation. With specified values for the r_i and the initial conditions, the problem becomes a division of one polynomial by another.

REFERENCES

1. L. E. Fuller. "Geometric Recurrence Relations." *The Fibonacci Quarterly* 18 (1980):126-129.
2. L. E. Fuller. "Representations for r, s Recurrence Relations." *The Fibonacci Quarterly* 18 (1980):129-135,
3. L. E. Fuller. "Solutions for General Recurrence Relations." *The Fibonacci Quarterly* 19 (1981):64-69.

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THE RESIDUES OF n^n MODULO p

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SUMMARY

In this paper we investigate the residues of $n^n \pmod p$, where $1 \le n \le p-1$ and p is an odd prime. We find new upper bounds for the number of distinct residues of $n^n \pmod p$ that can occur. We also give lower bounds for the number of quadratic nonresidues and primitive roots modulo p that do not appear among the residues of $n^n \pmod p$. Further, we prove that given any arbitrarily large positive integer M, there exist sets of primes $\{p_i\}$ and $\{q_j\}$, both with positive density in the set of primes, such that the congruences

$$x^x \equiv 1 \pmod{p_i},\ 1 \le x \le p_i - 1 \tag{1}$$

and

$$x^x \equiv -1 \pmod{q_j},\ 1 \le x \le q_j - 1 \tag{2}$$

both have at least M solutions.

1. INTRODUCTION

Roger Crocker [4] and [5] first examined the residues of n^n modulo p. It is clear that if $n \geq 1$, then the sequence $\{n^n\}$ reduced modulo p is periodic with a period of $p(p - 1)$. This follows from the facts that $(p-1, p) = 1$ and that if

$$n_1 \equiv n_2 \pmod{p} \text{ and } n_1 \equiv n_2 \pmod{p - 1},$$

then

$$n_1^{n_1} \equiv n_2^{n_2} \pmod{p}.$$

The following theorem shows that every residue appears among the residues of n^n modulo p, where $1 \leq n \leq p(p - 1)$, and counts the number of times a particular residue occurs.

Theorem 1: Consider the residues of n^n modulo p, where $1 \leq n \leq p(p - 1)$. Then the residue 0 appears $p - 1$ times. If $r \not\equiv 0 \pmod{p}$ and the exponent of r modulo p is d, then the number of times the residue r appears is

$$\sum_{d|d'|p-1} \phi(d')((p - 1)/d'). \tag{3}$$

Proof: First, it is clear that the residue 0 appears $p - 1$ times. Now consider any fixed nonzero residue n. It is raised to the various powers $n + kp$, where $0 \leq k \leq p - 2$. These powers form a complete residue system modulo $p - 1$. Thus n is raised to each power m, where $1 \leq m \leq p - 1$. Now, the congruence

$$n^x \equiv r \pmod{p} \tag{4}$$

is solvable for x if and only if

$$(p - 1, \text{Ind } n) \mid (p - 1, \text{Ind } r), \tag{5}$$

where Ind a is the index of a (mod p) with respect to a fixed primitive root. This can occur only if

$$\frac{p - 1}{(p - 1, \text{Ind } r)} \Bigg| \frac{p - 1}{(p - 1, \text{Ind } n)}. \tag{6}$$

but

$$\frac{p - 1}{(p - 1, \text{Ind } r)} = d$$

is the exponent of r (mod p) and

$$\frac{p - 1}{(p - 1, \text{Ind } r)} = d'$$

is the exponent of n (mod p). Thus congruence (4) has solutions if and only if d divides d'. It is evident that the number of solutions to (4) is then

$$(p - 1/d').$$

However, there are exactly $\phi(d')$ residues belonging to the expondnt d' (mod p). The theorem now follows.

From here on, we restrict n so that $1 \leq n \leq p - 1$. Then not every nonzero residue of p can appear among the residues of n^n (mod p). This follows from the fact that the residue 1 appears at least twice, since

$$1^1 \equiv 1 \quad \text{and} \quad (p - 1)^{p-1} \equiv 1 \pmod{p}.$$

We shall now address ourselves to determining how many and what types of residues modulo p can appear among the residues of n^n (mod p), where $1 \leq n \leq p - 1$.

2. A NEW UPPER BOUND FOR THE NUMBER OF DISTINCT RESIDUES OF n^n

Let $A(p)$ be the number of distinct residues of n^n (mod p), $1 \le n \le p - 1$. Roger Crocker [5] showed that

$$\sqrt{(p-1)/2} \le A(p) \le p - 4.$$

We obtain a much better upper bound for $A(p)$ in the following theorem.

Theorem 2: Let p be an odd prime. Let $A(p)$ be the number of distinct residues of n^n (mod p), where $1 \le n \le p - 1$. Then

$$A(p) < 3p/4 + C_1(\varepsilon)p^{1/2+\varepsilon}$$

where ε is any positive real number and $C_1(\varepsilon)$ is a constant depending solely on ε.

To establish Theorem 2 we shall estimate the number $N(p)$ of quadratic nonresidues not appearing among the residues of n^n (mod p), where $1 \le n \le p - 1$. We will in fact show that

$$N(p) > p/4 + C_2(\varepsilon)p^{1/2+\varepsilon} \tag{7}$$

where ε is any positive real number and $C_2(\varepsilon)$ is a constant dependent only on ε. It is easily seen that Theorem 2 then immediately follows.

The only way that n^n, $1 \le n \le p - 1$, can be a quadratic nonresidue is if n is odd. However, if n is odd and n is a quadratic residue, then n^n is not a quadratic nonresidue of p. Let $N_1(p)$ be the number of odd quadratic residues modulo p. Then

$$N(p) \ge N_1(p), \tag{8}$$

since the number of odd integers in the interval $(0, p)$ and the number of quadratic nonresidues are both equal to $(p-1)/2$. We refine inequality (8) by the following lemma.

Lemma 1: Let p be an odd prime. Let $1 \le n \le p - 1$. Let $N_1(p)$ be the number of integers in the interval $(0, p)$ for which n is an odd quadratic residue modulo p.

(i) At least $N_1(p)$ quadratic nonresidues do not appear among the residues of n^n (mod p).

(ii) If $p > 5$ and $p \equiv 5$ (mod 8) or $p > 7$ and $p \equiv 7$ (mod 8), then at least $N_1(p) + 1$ quadratic nonresidues (mod p) do not appear.

Proof: The proof of (i) follows from our discussion preceding the lemma. To prove (ii), first assume $p \equiv 5$ (mod 8). Then $(p + 1)/2$ and $p - 2$ are both odd quadratic nonresidues. Now, using Euler's criterion

$$((p+1)/2)^{(p+1)/2} \equiv 1^{(p+1)/2}/2^{(p+1)/2} \equiv 1/(2)2^{(p-1)/2} \equiv -1/2 \pmod{p}. \tag{9}$$

Also

$$(p-2)^{p-2} \equiv (-2)^{p-1}/(-2) \equiv -1/2 \pmod{p}. \tag{10}$$

Thus the quadratic nonresidues $((p + 1)/2)^{(p+1)/2}$ and $(p - 2)^{p-2}$ are identical. Now n^n can be a quadratic nonresidue only if n is already a quadratic nonresidue (in fact odd) and two such residues repeat. Thus, by part (i), at least $N_1(p) + 1$ residues do not appear among $\{n^n\}$ modulo p, where $1 \le n \le p - 1$.

Now suppose $p \equiv 7$ (mod 8). Then $(3p - 1)/4$ and $p - 2$ are both odd quadratic nonresidues modulo p. Further,

$$\begin{aligned}((3p-1)/4)^{(3p-1)/4} &\equiv -1/4^{(3p-1)/4} \equiv -1/2^{(3p-1)/2} \\ &\equiv -1/2^{3((p-1)/2)+1} \equiv -1/2 \pmod{p}.\end{aligned} \tag{11}$$

Again,

$$(p-2)^{p-2} \equiv -1/2 \pmod{p}. \tag{12}$$

The result now follows as before.

According to Lemma 1, we now need a determination of $N_1(p)$ to establish an upper bound for $A(p)$. Lemmas 2, 3, and 4 will provide this information.

Lemma 2: If $p \equiv 1 \pmod 4$, then $N_1(p) = (p - 1)/4$.

Proof: Let r be a quadratic nonresidue modulo p. Then $p - r$ is also a quadratic nonresidue. But exactly one of p and $p - r$ is odd. Hence exactly half of the $(p - 1)/2$ quadratic nonresidues of p are odd and $N_1(p) = (p - 1)/4$.

Lemma 3: If $p \equiv 7 \pmod 8$, then $N_1(p) = (p - 1 - 2h(-p))/4$, where $h(-p)$ is the class number of the algebraic number field $Q(\sqrt{-p})$.

Proof: It is known (see [3]) that

$$h(-p) = V - T,$$

where V and T denote the number of quadratic residues and quadratic nonresidues in the interval $(0, p/2)$, respectively. To evaluate $V - T$, we will make use of the sum of Legendre symbols

$$S = \sum_{0<n<p/2} (n/p).$$

We partition S in two different ways as

$$S = S_1 + S_2 = S' + S'', \tag{13}$$

where

$$S_1 = \sum_{0<n<p/4} (n/p), \quad S_2 = \sum_{p/4<n<p/2} (n/p)$$

$$S' = \sum_{\substack{0<n<p/2 \\ n \text{ even}}} (n/p), \quad S'' = \sum_{\substack{0<n<p/2 \\ n \text{ odd}}} (n/p).$$

It is known (see [2]) that $S_2 = 0$. Then

$$S = S' + S'' = (2/p) \sum_{0<j<p/4} (j/p) + S'' = (1)S_1 + S'' = S_1 + S_2. \tag{14}$$

Hence $S'' = S_2 = 0$.

Now let V_o and T_o denote the number of odd quadratic residues and nonresidues in $(0, p/2)$, respectively. Let V_e and T_e be the number of even quadratic residues and nonresidues in $(0, p/2)$, respectively. Inspection shows that

$$V_o + T_o = (p + 1)/4 \quad \text{and} \quad V_e + T_e = (p - 3)/4.$$

Since $S'' = 0$,

$$V_o = T_o = (p + 1)/8. \tag{15}$$

Further,

$$h(-p) = V - T = (V_o - T_o) + (V_e - T_e) = V_e - T_e. \tag{16}$$

Also,

$$(p - 3)/4 = V_e + T_e. \tag{17}$$

Solving (16) and (17) for T_e, we obtain

$$T_e = (p - 3 - 4h(-p))/8.$$

Finally,

$$N_1(p) = V_o + T_e = (p - 1 - 2h(-p))/4,$$

since the number of odd quadratic residues in $(p/2, p)$ equals the number of even quadratic nonresidues in $(0, p/2)$.

Lemma 4: If $p \equiv 3 \pmod 8$, then $N_1(p) = (p - 1 + 6h(-p))/4$.

Proof: We shall use the same notation as in the proof of Lemma 3. By [3],

$$h(-p) = 1/3(V - T).$$

As in the proof of Lemma 3, we now evaluate the sum of Legendre symbols S.

$$S = S' + S'' = (2/p) \sum_{0<j<p/4} (j/p) + S'' = (-1)S_1 + S'' = S_1 + S_2. \tag{18}$$

However, it is known (see [2]) that $S_1 = 0$. Hence, $S_2 = S'' = S$ and $S_1 = S'$. Examination shows that

$$V_e + T_e = (p - 3)/4 \quad \text{and} \quad V_o + T_o = (p + 1)/4.$$

Since $S' = 0$,

$$V_e = T_e = (p - 3)/8. \tag{19}$$

Thus,

$$h(-p) = (1/3)(V - T) = (1/3)[(V_o - T_o) + (V_e - T_e)] = (1/3)(V_o - T_o) \tag{20}$$

and

$$(p + 1)/4 = V_o + T_o.$$

Solving (20) and (21) for V_o, we obtain

$$V_o = (12h(-p) + p + 1)/8.$$

Hence

$$N_1(p) = V_o + T_e = (p - 1 + 6h(-p))/4.$$

We utilize our results of Lemmas 1-4 in estimating $N(p)$ in the following theorem.

Theorem 3: Let p be an odd prime. Let $N(p)$ be the number of quadratic nonresidues not appearing among the residues n^n, where $1 \le n \le p - 1$.

(i) $N(p) \ge (p - 1)/4$ if $p \equiv 1 \pmod 8$.
(ii) $N(p) \ge (p + 3)/4$ if $p > 5$ and $p \equiv 5 \pmod 8$.
(iii) $N(p) \ge (p - 1 + 6h(-p))/4$ if $p \equiv 3 \pmod 8$.
(iv) $N(p) \ge (p + 3 - 2h(-p))/4$ if $p > 7$ and $p \equiv 7 \pmod 8$.

Proof: This follows from Lemmas 1-4.

We are now ready for the proof of our main theorem.

Proof of Theorem 2: By Siegel's theorem [1],

$$h(-p) < C_2(\varepsilon)p^{1/2+\varepsilon}$$

where ε is a positive real number and $C_2(\varepsilon)$ is a constant dependent solely on ε. Note that

$$A(p) \le p - 1 - N(p).$$

The theorem now follows from Theorem 3.

3. PRIMITIVE ROOTS NOT APPEARING AMONG THE RESIDUES OF n^n

In Section 2 we determined lower bounds for the number of quadratic nonresidues not appearing among the residues of n^n modulo p. In this section we determine lower bounds for the number of primitive roots (mod p) that do not appear among the residues of n^n (mod p), where $1 \le n \le p - 1$. Crocker [4] has shown that n^n can be congruent to a primitive root (mod p) only if $(n, p - 1) = 1$, where $1 \le n \le p - 1$. Using this criterion, we shall prove Theorem 4.

Theorem 4: Let p be an odd prime. Let $1 \le n \le p - 1$.

(i) At least one primitive root does not appear among the residues of n^n (mod p).

(ii) If $p \equiv 1 \pmod 8$ or $p \equiv 3 \pmod 8$ and $p > 3$, then at least three primitive roots do not appear among the residues of n^n (mod p).

Proof of (i): Note that 1^1 is not congruent to a primitive root (mod p). Now, n^n can be congruent to a primitive root (mod p) only if $(n, p - 1) = 1$. Certainly $(1, p - 1) = 1$. Hence at least one primitive root does not appear among the residues of n^n (mod p), since n^n can be a primitive root only if n already is.

Proof of (ii): Suppose $p \equiv 1 \pmod 8$. Then $((p + 1)/2, p - 1) = 1$. But

$$((p + 1)/2)^{(p+1)/2} \equiv 1/2 \pmod p. \tag{22}$$

However, $(2^{-1}/p) = 1$. Thus $((p + 1)/2)^{(p+1)/2}$ is not congruent to a primitive root (mod p). Also, $(p - 2, p - 1) = 1$ and

$$(p - 2)^{p-2} \equiv -1/2 \pmod p. \tag{23}$$

Again, $((-2)^{-1}/p) = 1$ and $(p - 2)^{p-2}$ is not congruent to a primitive root (mod p). Hence at least three primitive roots do not appear.

Now suppose $p \equiv 3 \pmod 8$. As before, $(p - 2, p - 1) = 1$, and $(p - 2)^{p-2}$ is not congruent to a primitive root (mod p). Further, $((p + 1)/4, p - 1) = 1$ and

$$((p + 1)/4)^{(p+1)/4} \equiv -1/2 \pmod p. \tag{24}$$

However, $((-2)^{-1}/p) = 1$ and consequently $((p + 1)/4)^{(p+1)/4}$ is not congruent to a primitive root (mod p). Thus at least three primitive roots do not appear among the residues of n^n (mod p) if $p \equiv 3 \pmod 8$ and $p > 3$.

4. THE NUMBER OF TIMES THE RESIDUES 1 AND -1 APPEAR

Theorems 5 and 6 in this section will show that there is no upper bound for the number of times that the residues 1 or -1 can appear among the residues of n^n (mod p), $1 \le n \le p - 1$, where p is allowed to vary among all the primes.

Theorem 5: Let M be any positive integer. Let $\{p_i\}$ be the set of primes such that

$$x^x \equiv 1 \pmod{p_i}, \tag{25}$$

where $1 \le x \le p_i - 1$, has at least M solutions. Then $\{p_i\}$ has positive density in the set of primes.

Proof: Let $N = M - 1$. Let $p \equiv 1 \pmod{2^N}$ be a prime. Suppose that 2 is a 2^Nth power (mod p). Then, if $0 \le k \le N - 1$, 2^k is a 2^kth power (mod p). Further, if $0 \le k \le N - 1$, $(p - 1)/2^k$ is an even integer. Now, if x is a dth power (mod p) and $p \equiv 1 \pmod d$, then

$$x^{(p-1)/d} \equiv 1 \pmod p.$$

Hence, if $0 \le k \le N - 1$,

$$((p - 1)/2^k)^{(p-1)/2^k} \equiv (-1)^{(p-1)/2^k}/(2^k)^{(p-1)/2^k} \equiv 1/1 \equiv 1 \pmod p. \tag{26}$$

Thus we now have M solutions to congruence (25); namely, 1 and $(p - 1)/2^k$ for $0 \le k \le N - 1$.

We now show that the set of primes p_i such that $p_i \equiv 1 \pmod{2^N}$ and 2 is a 2^Nth power (mod p) indeed has positive density t in the set of primes. Let ζ be a primitive 2^Nth root of unity. Let L be the algebraic number field

$$Q(2^{1/2^N}, \zeta).$$

Let $p \equiv 1 \pmod{2^N}$ be a rational prime. Suppose that 2 is a 2^Nth power (mod p).

By Kummer's theorem, this occurs if and only if, in the field L, p splits completely in each of the subfields $Q(\zeta^k \cdot 2^{1/2^N})$, where $1 \le k \le 2^N$. Let P be a prime ideal of L dividing the principal ideal (p). Let Z_P be the decomposition field of P. Then $Z_P \supseteq Q(\zeta^k \cdot 2^{1/2^N})$ for $1 \le k \le 2$, since p splits completely in each of these subfields. Hence $Z_P \supseteq Q(\zeta, 2^{1/2^N}) = L$, the compositum of the subfields $Q(\zeta^k \cdot 2^{1/2^N})$, where $1 \le k \le 2^N$. Let D_P be the decomposition group of P. Then $D_P = \langle 1 \rangle$ for all prime ideals P dividing (p). Thus, by the Tchebotarev density theorem, the density

$$t = 1/[L:Q] = 1/2^{2N-2} = 1/2^{2M-4} > 0. \tag{27}$$

Theorem 6: Let M be any positive integer. Let $\{p_i\}$ be the set of primes such that the congruence

$$x^x \equiv -1 \pmod{p_i}, \tag{28}$$

where $1 \le x \le p_i - 1$, has at least M solutions. Then $\{p_i\}$ has positive density in the set of primes.

Proof: Let $N = M - 1$. Let p be a prime and suppose that $p \equiv 1 \pmod{2 \cdot 3^N}$ and $p \equiv 7 \pmod 8$. Suppose further that both 2 and 3 are $(2 \cdot 3^N)$th powers (mod p). Note that if $p \equiv 7 \pmod 8$, $(2/p) = 1$, and it is possible that 2 is a $(2 \cdot 3^N)$th power (mod p). Then, if $1 \le k \le N$, $2 \cdot 3^k$ is a $(2 \cdot 3^k)$th power (mod p). Moreover, if $1 \le k \le N$, $(p - 1)/(2 \cdot 3^k)$ is an odd integer. Hence, if $1 \le k \le N$,

$$\begin{aligned}((p - 1)/(2 \cdot 3^k))^{(p-1)/(2\cdot 3^k)} &\equiv (-1)^{(p-1)/(2\cdot 3^k)} / (2 \cdot 3\ )^{(p-1)/(2\cdot 3^k)} \\ &\equiv -1/1 \equiv -1 \pmod p.\end{aligned} \tag{29}$$

Thus we now have M solutions to congruence (28).

I now claim that the set of primes $\{p_i\}$ such that $p_i \equiv 1 \pmod{2 \cdot 3\ }$, $p_i \equiv 7 \pmod 8$, and both 2 and 3 are $(2 \cdot 3^N)$th powers (mod p_i) has positive density u in the set of primes. Let ζ be a primitive $(4 \cdot 3^N)$th root of unity. Let L be the algebraic number field

$$Q(\zeta, 2^{1/(2\cdot 3^N)}, 3^{2/(2\cdot 3^N)}).$$

Suppose that p is a rational prime and that $p \equiv 1 \pmod{2 \cdot 3^N}$ and $p \equiv 7 \pmod 8$. Assume that both 2 and 3 are $(2 \cdot 3^N)$th powers (mod p). Then, by Kummer's theorem, p splits completely in each of the subfields

$$Q(\zeta^{2k} \cdot 2^{1/(2\cdot 3^N)}) \quad\text{and}\quad Q(\zeta^{2k} \cdot 3^{1/(2\cdot 3^N)}),$$

where $1 \le k \le 2 \cdot 3^N$. Hence p splits completely in

$$K = Q(\zeta^2, 2^{1\ (2\cdot 3^N)}, 3^{1\ (2\cdot 3^N)}),$$

the compositum of these subfields. Let P be a prime ideal in L dividing (p). Then, if Z_P is the decomposition field of P, $Z_P \supseteq K$. Furthermore, since $p \equiv 7 \pmod 8$, $(-1/p) = -1$, and p does not split in the subfield $Q(\sqrt{-1})$ of L. Consequently, $Z_P \not\supseteq Q(\sqrt{-1})$. Let σ be the automorphism of Gal (L/Q) such that

$$\sigma(\zeta) = -\zeta, \tag{30}$$

$$\sigma(2^{1/(2\cdot 3^N)}) = 2^{1/(2\cdot 3^N)},$$

and

$$\sigma(3^{1/(2\cdot 3^N)}) = 3^{1/(2\cdot 3^N)}.$$

Then $\langle\sigma\rangle$ is the subgroup of Gal (L/Q) fixing K. It follows that the decomposition group $D_P = \langle\sigma\rangle$ for all prime ideals P dividing (p). By the Tchebotarev density theorem, the density

$$u = 1/[L:Q] = 1/(8 \cdot 3^{3N-1}) = 1/(8 \cdot 3^{3M-4}) > 0. \tag{31}$$

5. CONCLUDING REMARK

Further problems concerning the residues of $n^n \pmod p$, where $1 \leq n \leq p-1$, are obtaining better upper and lower bounds for the number of distinct residues appearing among $\{n^n\}$ and determing estimates for the number of times that residues other than ±1 may occur.

REFERENCES

1. Raymond Ayoub. *An Introduction to the Analytic Theory of Numbers*. Mathematical Surveys No. 10. Providence, R.I.: American Mathematical Society, 1963. Pp. 327-342.
2. Bruce C. Berndt & S. Chowla. "Zero Sums of the Legendre Symbol." *Nordisk Matematisk Tidskrift* 22 (1974):5-8.
3. Z. I. Borevich & I. R. Shafarevich. *Number Theory*. New York: Academic Press, 1066. Pp. 346-347,
4. Roger Crocker. "On a New Problem in Number Theory." *Amer. Math. Monthly* 73 (1966):355-357.
5. Roger Crocker. "On Residues of n^n." *Amer. Math. Monthly* 76 (1969):1028-1929.

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A GENERALIZATION OF A PROBLEM OF STOLARSKY

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For fixed positive integer $k \geq 1$, we set

$$a_k = [\overline{k}] = \frac{k + \sqrt{k^2 + 4}}{2},$$

a real number with completely periodic continued fraction expansion and period of length one. For all integers $n \geq 1$, we use $f_k(n)$ to denote the nearest integer to na_k.

Using this notation, we define an array $(b_{i,j}^{(k)})$ as follows. The first row has

$$b_{1,1}^{(k)} = 1 \quad \text{and} \quad b_{1,j}^{(k)} = f\ (b_{1,j-1}^{(k)}), \text{ for all } j \geq 2.$$

After inductively setting $b_{i,1}^{(k)}$ to be the smallest integer that has not occurred in a previous row, we define the remainder of the ith row by

$$b_{i,j}^{(k)} = f_k(b_{i,j-1}^{(k)}), \text{ for all } j \geq 2.$$

K. Stolarsky [4] developed this array for $k = 1$, showed that each positive integer occurs exactly once in the array, and proved that any three consecutive entries of each row satisfy the Fibonacci recursion. The latter result can be viewed as a generalization of a result of V. E. Hoggatt, Jr. [3, Theorem III]. In Theorem 1, we prove an analogous result for general k.

Theorem 1: Each positive integer occurs exactly once in the array $(b_{i,j}^{(k)})$. Moreover, the rows of the array satisfy

$$b_{i,j+2}^{(k)} = kb_{i,j+1}^{(k)} + b_{i,j}^{(k)}, \text{ for all } i, j \geq 1.$$

Proof: By construction, each positive integer occurs at least once. For $m \neq n$ we have $|(n - m)a_k| > 1$ and so $f_k(m) \neq f_k(n)$. Since the first column entry is the smallest in any row, every positive integer occurs exactly once.