The great geographical distance between us prevented us from seeing one another very often. I did, on my way to lecturing in Hawaii, stop off to see Vern, and I spent a few days with him a couple of years later when I lectured along the California coast. He once visited me at the University of Maine, when, representing his university, he came as a delegate to a national meeting of Phi Kappa Phi (an academic honorary that was founded at the University of Maine). For almost four decades I had the enormous pleasure of Vern's friendship, and bore the flattering title, generously bestowed upon me by him, of his "mathematical mentor."

In mathematics, Vern was a skylark, and I regret, far more than I can possibly express, the sad fact that we now no longer will hear further songs by him. But, oh, on the other hand, how privileged I have been; I heard the skylark when he first started to sing.

Hail to thee, blithe Spirit!
Bird thou never wert,
That from Heaven, or near it,
Pourest thy full heart
In profuse strains of unpremeditated art.

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DIAGONAL SUMS IN THE HARMONIC TRIANGLE

MARJORIE BICKNELL-JOHNSON
A. C. Wilcox High School, Santa Clara CA 95051

Dedicated to the memory of my colleague and friend, Verner Hoggatt

Leibniz's harmonic triangle is related to reciprocals of the elements of Pascal's triangle, and was developed in summing infinite series by a telescoping process as discussed by Kneale [1] and Price [2], among others. Here, we find row sums and rising diagonal sums for the harmonic triangle.

1. PROPERTIES OF THE HARMONIC TRIANGLE

The harmonic triangle of Leibniz

$$\begin{array}{ccccccccccccccc}
\frac{1}{1} & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \frac{1}{6} & & \frac{1}{7} & \cdots \\
& \frac{1}{2} & & \frac{1}{6} & & \frac{1}{12} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{42} & \cdots & \\
& & \frac{1}{3} & & \frac{1}{12} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{105} & \cdots & & \\
& & & \frac{1}{4} & & \frac{1}{20} & & \frac{1}{60} & & \frac{1}{140} & \cdots & & & \\
& & & & \frac{1}{5} & & \frac{1}{30} & & \frac{1}{105} & \cdots & & & & \\
& & & & & \frac{1}{6} & & \cdots & & \cdots & & & & &
\end{array}$$

is formed by taking successive differences of terms of the harmonic series.

After the first row, each entry is the difference of the two elements immediately above it, as well as being the sum of the infinite series formed by the entries in the row below and to the right. Also, each element is the sum of the element to its right and the element below it in the array. For example, for 1/6 circled above, 1/2 - 1/3 = 1/6, and

$$\begin{aligned}\frac{1}{6} &= \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \cdots \\ &= \left(\frac{1}{6} - \frac{1}{12}\right) + \left(\frac{1}{12} - \frac{1}{20}\right) + \left(\frac{1}{20} - \frac{1}{30}\right) + \left(\frac{1}{30} - \frac{1}{42}\right) + \cdots\end{aligned}$$

Notice that each row has the first element in the row above it as its sum.

Each rising diagonal contains elements which are $1/n$ times the reciprocal of the similarly placed elements in Pascal's triangle

$$\begin{array}{ccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \ldots \\
& 1 & & 2 & & 3 & & 4 & & 5 & & 6 & \ldots & \\
& & 1 & & 3 & & 6 & & 10 & & 15 & \ldots & & \\
& & & 1 & & 4 & & 10 & & 20 & \ldots & & & \\
& & & & 1 & & 5 & & 15 & \ldots & & & & \\
& & & & & 1 & & 6 & \ldots & & & & & \\
& & & & & & \ldots & \ldots & & & & & &
\end{array}$$

In contrast to the harmonic triangle, each element in any row after the first is the sum of all terms in the row above it and to the left, while it is also the difference of the two terms in the row beneath it, and the sum of the element to its left and the element above it.

Since the nth row in the harmonic triangle has sum $1/(n - 1)$, if we multiply the row by n, we can immediately write the sum of the reciprocals of elements found in the columns of Pascal's triangle written in left-justified form as

$$\frac{n}{n-1} = \sum_{i=n}^{\infty} \binom{i}{n}^{-1}, \; n > 1, \tag{1.1}$$

or we can begin by summing after k terms, as

$$\frac{n+1}{n} \cdot \binom{n+k}{n}^{-1} = \sum_{i=n+1}^{\infty} \binom{i+k}{n+1}^{-1} \tag{1.2}$$

As a corollary, we can easily sum the reciprocals of the triangular numbers $T = n(n + 1)/2$ by taking $n = 2$ in (1.1), or we could simply multiply the second row of the harmonic triangle by 2.

2. ROW SUMS OF THE HARMONIC TRIANGLE

We rewrite the harmonic triangle in left-justified form as

1/1					
1/2	1/2				
1/3	1/6	1/3			
1/4	1/12	1/12	1/4		
1/5	1/20	1/30	1/20	1/5	
1/6	1/30	1/60	1/60	1/30	1/6
...	...	...	...	...	...

We number the rows and columns to begin with zero. Then the element in the ith row and nth column is

$$1/\left[(i+1)\binom{i}{n}\right], \quad n = 0, 1, 2, \ldots, \quad i = 0, 1, 2, \ldots .$$

The row sums are 1, 1, 5/6, 2/3, 8/15, 13/30, 151/420, ..., which sequence is the convolution of the harmonic sequence 1, 1/2, 1/3, ..., $1/n$, ..., and the sequence 1, 1/2, 1/4, 1/8, ..., $1/2^n$, ..., which can be derived [3] as follows.

Let $G_n(x)$ be the generating function for the elements in the nth column of the harmonic triangle written in left-justified form. Then

$$G_0(x) = \ln[1/(1-x)] = 1 + x/2 + x^2/3 + \cdots + x^n/(n+1) + \cdots \tag{2.1}$$

and generally,

$$G_{n+1}(x) = (x-1)G_n(x) + x^n/(n+1). \tag{2.2}$$

Consider the display

$$\begin{aligned}
G_0(x) &= & & G_0(x) \\
G_1(x) &= (x-1)G_0(x) &+& 1 \\
G_2(x) &= (x-1)G_1(x) &+& x/2 \\
\ldots & \quad\ldots & & \ldots \\
G_{n+1}(x) &= (x-1)G_n(x) &+& x^n/(n+1) \\
\ldots & \quad\ldots & & \ldots
\end{aligned}$$

Let S be the infinite sum of the column generators, and sum vertically:

$$S = (x-1)S + 2G_0(x).$$

Solving for S, we have

$$S = G_0(x)/(1-x/2) = \left(\ln[1/(1-x)]\right)\cdot\left(\frac{1}{1-x/2}\right),$$

the product of the generating functions for the harmonic sequence and for the sequence of powers of 1/2. Thus, the row sums are the convolution between the harmonic sequence $\left\{1/n\right\}_{n=1}^{\infty}$ and the sequence $\left\{1/2^n\right\}_{n=0}^{\infty}$.

What we have found is

$$\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}^{-1}\right)x^n = \frac{\ln[1/(1-x)]}{1-x/2}, \tag{2.3}$$

$$\frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}^{-1} = \sum_{k=0}^{n}\frac{1}{k+1}\cdot\frac{1}{2^{n-k}} \tag{2.4}$$

We can also write the generating function $S^*(x)$ for the sums of elements appearing on the successive rising diagonals formed by beginning in the leftmost column and proceeding up p elements and right one element throughout the array:

$$S^*(x) = \frac{G_0(x) + x^pG_0(x^{p+1})}{1 + x^p - x^{p+1}} \tag{2.5}$$

By way of comparison, the Fibonacci numbers with negative subscripts are generated by

$$\frac{1}{1 + x - x^2} = \sum_{n=1}^{\infty} F_{-n} x^{n-1}. \tag{2.6}$$

To contrast with the harmonic triangle, we write Pascal's triangle in left-justified form, and number the rows and columns to begin with zero:

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
...	...	...	...	...	...	...

Then the generating function for the nth column is $G_n(x) = x^n/(1-x)^{n+1}$, which has been used to write the diagonal sums for Pascal's triangle [4], [7]. We recall the numbers $u(n;p,q)$ of Harris and Styles [5], [6], formed as the sum of the element in the leftmost column and nth row and the elements obtained by taking steps p units up and q units right throughout the array. These numbers are generated by [4], [7],

$$\sum_{n=0}^{\infty} u(n;p,q)x = \frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}}, \; p + q \geq 1, \; q \geq 0. \tag{2.7}$$

Also, the row sums of Pascal's triangle are given by $2^n = u(n;0,1)$, while the Fibonacci numbers are the sums on the rising diagonals, or, $u(n;1,1) = F_n$. If we extend $u(n;p,q)$ to negative subscripts [3], we have

$$\frac{1}{1 + x^p - x^{p+1}} = \sum_{n=0}^{\infty} u(-n;p,1)x^n, \tag{2.8}$$

which has a form similar to (2.5) and becomes (2.6) when $p = 1$.

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