SOME GENERALIZATIONS OF A BINOMIAL IDENTITY CONJECTURED BY HOGGATT

L. CARLITZ
Duke University, Durham NC 27706

To the memory of Verner Hoggatt

1. INTRODUCTION

In November 1979 Hoggatt sent me the following conjectured identity. Put

$$S_{n,r} = \frac{1}{r+1}\binom{n-r}{r}\binom{n-r-1}{r} \quad (n \geq 2r+1;\ r \geq 0). \tag{1.1}$$

Then

$$S_{n+1,r} = S_{n,r} + \sum_{k=0}^{r-1}\sum_{j=1}^{n-1} S_{j,k}S_{n-j-1,r-k-1} \quad (n \geq 2r+1;\ r \geq 1). \tag{1.2}$$

I was able to send him a proof of (1.2) that made use of various properties of special functions.

In this note we first sketch this proof. Next, using a different method, we obtain some generalizations of (1.2). In particular, if we put

$$S_{n,r}^{p} = \frac{1}{r+1}\binom{n-r}{r}\binom{n-r-p}{r} \quad (n \geq 2r+p;\ r \geq 0), \tag{1.3}$$

where p is a nonnegative integer, we show that

$$(p+q)S_{n,r}^{(p+q)} = pS_{n-q,r}^{(p)} + qS_{n-p,r}^{(q)} + pq\sum_{j=0}^{n-2}\sum_{s=0}^{r-1} S_{j,s}^{(p)}S_{n-j-2,r-s-1}^{(q)} \tag{1.4}$$

$$(p > 0,\ q > 0,\ r > 0)$$

and

$$(r+1)S_{n,r}^{(p+q)} = (r+1)S_{n,r}^{(q)} + p\sum_{j=0}^{n-2}\sum_{s=0}^{r-1}(r-s)S_{j,s}S_{n-j-2,r-s-1} \tag{1.5}$$

$$(p > 0,\ q \geq 0,\ r > 0).$$

We remark that (1.4) is implied by (1.5).

The special case $p = 1$, $q = 0$, of (1.5) may be noted:

$$(r+1)S_{n,r} = \binom{n-r}{r}^2 + \sum_{j=0}^{n-2}\sum_{s=0}^{r-1}\binom{j-s}{s}^2 S_{n-j-2,r-s-1}. \tag{1.6}$$

For additional results, see (7.7) and (7.8) below.

Remark: The close relationship between the identities of this paper and ultraspherical polynomials suggests that even more general identities can be found that are related to the general Jacobi polynomials. This is indeed the case; however, we leave this for another paper.

SECTION 2

Put

$$f_r(x) = \frac{1}{r+1}\sum_{n=2r+1}^{\infty}\binom{n-r}{r}\binom{n-r-1}{r}x^n = \sum_{n=2r+1}^{\infty} S_{n,r}x^n. \tag{2.1}$$

It follows that

$$f_r(x) = \frac{x^{2r+1}}{(1-x)^{2r+1}}\phi_r(x), \tag{2.2}$$

where $\phi_r(x)$ is a polynomial in x. To get an explicit formula for $\phi_r(x)$, rewrite (2.2) in the form

$$x^{2r+1}\phi_r(x) = (1-x)^{2r+1}f_r(x).$$

Thus

$$\begin{aligned}
\phi_r(x) &= \frac{1}{r+1}(1-x)^{2r+1}\sum_{n=0}^{\infty}\binom{n+r}{r}\binom{n+r+1}{r}x^n \\
&= \frac{1}{r+1}\sum_{j=0}^{2r+1}(-1)^j\binom{2r+1}{j}x^j\sum_{n=0}^{\infty}\binom{n+r}{r}\binom{n+r+1}{r}x^n \\
&= \frac{1}{r+1}\sum_{m=0}^{\infty}x^m\sum_{\substack{j=0\\ j\le m}}^{2r+1}(-1)^j\binom{2r+1}{j}\binom{m-j+r}{r}\binom{m-j+r+1}{r}.
\end{aligned} \tag{2.3}$$

Since the product

$$\binom{m-j+r}{r}\binom{m-j+r+1}{r}$$

is of degree $2r$ in j, it follows that the inner sum in (2.3) vanishes for

$$m \ge 2r+1.$$

Thus we need only consider $m \le 2r$. Hence the sum is equal to

$$\binom{m+r}{r}\binom{m+r+1}{r}\sum_{j=0}^{m}\frac{(-2r-1)_j(-m)_j(-m-1)_j}{j!(-m-r)_j(-m-r-1)_j},$$

where

$$(a)_j = a(a+1)\ \dots\ (a+j-1).$$

Applying Saalschütz' theorem [1, p. 87], we get

$$\binom{m+r}{r}\binom{m+r+1}{r}\frac{(-r+1)_m(-m+r+1)_m}{(-m-r)_m(r+2)_m} = \frac{r+1}{m+1}\binom{r}{m}\binom{r-1}{m}.$$

We have, therefore,

$$\phi_r(x) = \sum_{m=0}^{r-1}\frac{1}{m+1}\binom{r}{m}\binom{r-1}{m}x^m \quad (r \ge 1). \tag{2.4}$$

For $r = 0$, it is clear that

$$\phi_0(x) = 1. \tag{2.5}$$

In hypergeometric notation, (2.4) becomes

$$\phi_r(x) = {}_2F_1[-r+1, -r; 1; x]. \tag{2.6}$$

On the other hand [1, p. 254, Eq. (2)],

$$P_n^{(1,1)}(x) = \frac{(2)_n}{n!}\left(\frac{x+1}{2}\right)^n {}_2F_1\left[-n, -n-1; 2; \frac{x-1}{x+1}\right].$$

If we put

$$y = \frac{x-1}{x+1},\ x = \frac{1+y}{1-y},$$

this becomes

$$_2F_1[-n, -n - 1; 2; y] = \frac{1}{n + 1}(1 - y)^n P_n^{(1,1)}\left(\frac{1 + y}{1 - y}\right).$$

Thus, by (2.6),

$$\phi_{r+1}(x) = \frac{1}{r + 1}(1 - x)^r P_r^{(1,1)}\left(\frac{1 + x}{1 - x}\right). \tag{2.7}$$

We have also the generating function [1, p. 271, Eq. (6)]

$$\sum_{n=0}^{\infty} P_n^{(1,1)}(x)t^n = 2^2\rho^{-1}(1 + t + \rho)^{-1}(1 - t + \rho)^{-1},$$

where

$$\rho = (1 - 2xt + t^2)^{1/2}.$$

Thus

$$(1 + t + \rho)(1 - t + \rho) = 2(1 - xt + \rho),$$

so that

$$\sum_{n=0}^{\infty} P^{(1,1)}(x)t^n = 2\rho^{-1}(1 - xt + \rho)^{-1}. \tag{2.8}$$

It can be verified that if

$$\Phi = \frac{1 - xt + \rho}{t},$$

then

$$\frac{d\Phi}{dt} = \frac{x^2 - 1}{\rho(1 - xt + \rho)}.$$

Comparison with (2.8) gives

$$\sum_{n=0}^{\infty} \frac{1}{n + 1}P_n^{(1,1)}(x)t^{n+1} = \frac{2}{x^2 - 1}\,\frac{1 - xt - \rho}{t}. \tag{2.9}$$

Now replace x by $(1 + x)/(1 - x)$ and replace t by $(1 - x)z$. The result is

$$\sum_{n=0}^{\infty} \frac{1}{n + 1}(1 - x)^n P_n^{(1,1)}\left(\frac{1 + x}{1 - x}\right)z^n = \frac{1 - (1 + x)z - \sqrt{1 - 2(1 + x)z + (1 - x)^2 z^2}}{2xz^2}.$$

Thus, by (2.7), we get

$$\sum_{r=0}^{\infty} \phi_{r+1}(x)z^r = \frac{1 - (1 + x)z - \sqrt{1 - 2(1 + x)z + (1 - x)^2 z^2}}{2xz^2}. \tag{2.10}$$

SECTION 3

We now rewrite the identity (1.2) in terms of the polynomial $\phi_r(x)$. To begin with, (1.2) can be replaced by

$$\begin{aligned} S_{n+1,r} &= S_{n,r} + \sum_{k=0}^{r-1}\sum_{j=1}^{n-2} S_{j,k}S_{n-j-1,r-k-1} + \sum_{k=0}^{r-1} S_{n-1,k}S_{0,r-k-1} \\ &= S_{n,r} + S_{n-1,r-1} + \sum_{k=0}^{r-1}\sum_{j=1}^{n-2} S_{j,k}S_{n-j-1,r-k-1}. \end{aligned}$$

Then multiplying both sides by x^{n+1} and summing over n we get

$$\sum_{n=2r+1}^{\infty} S_{n+1,r}x^{n+1} = x\sum_{n=2r+1}^{\infty} S_{n,r}x^n + x^2\sum_{n=2r}^{\infty} S_{n,r-1}x^n$$
$$+ x^2\sum_{k=0}^{r-1}\sum_{j=2k+1}^{\infty} S_{j,k}x^j \sum_{n=2r-2k-1}^{\infty} S_{n,r-k-1}x^n.$$

In view of (2.1), this becomes

$$(1 - x)f_r(x) = x^2 f_{r-1}(x) + x^2\sum_{k=0}^{r-1} f_k(x)f_{r-k-1}(x).$$

Hence by (2.2) we get

$$\phi_r(x) = (1 - x)\phi_{r-1}(x) + x\sum_{k=0}^{r-1}\phi_k(x)\phi_{r-k-1}(x) \quad (r \geq 1). \tag{3.1}$$

For example, we have

$$\phi_1(x) = 1,\ \phi_2(x) = 1 + x,\ \phi_3(x) = 1 + 3x + x^2,\ \phi_4(x) = 1 + 6x + 6x^2 + x^3$$

in agreement with (2.4).

Next put

$$F = F(x, z) = \sum_{r=0}^{\infty}\phi_r(x)z^r;$$

then it is easily verified that (3.1) gives

$$F = 1 + (1 - x)zF + xzF^2. \tag{3.2}$$

The solution of (3.2) such that $F(x, 0) = 1$ is

$$F = \frac{1 - (1 - x)z - \sqrt{1 - 2(1 + x)z + (1 - x)^2z^2}}{2xz}.$$

Since

$$\frac{F - 1}{z} = \sum_{r=0}^{\infty}\phi_{\quad 1}(x)z^r,$$

we get

$$\sum_{r=0}^{\infty}\phi_{r+1}(x)z^r = \frac{1 - (1 + x)z - \sqrt{1 - 2(1 + x)z + (1 - x)^2z^2}}{2xz^2}. \tag{3.3}$$

Comparison of (3.3) with (2.10) evidently completes the proof of the desired result.

SECTION 4

To generalize the above, we take

$$S_{n,r}^{(p)} = \frac{1}{r + 1}\binom{n - r}{r}\binom{n - r - p}{r} \tag{4.1}$$

and

$$f_r^{(p)}(x) = \sum_{n=2r+p}^{\infty} S_{n,r}^{(p)}x^n, \tag{4.2}$$

where p is a fixed nonnegative integer. Clearly

$$f_r^{(p)}(x) = \frac{x^{2r+p}}{(1 - x)^{2r+1}}\phi_r^{(p)}(x), \tag{4.3}$$

where $\phi_r^{(p)}(x)$ is a polynomial x. It is evident that

$$S_{n,r} = S_{n,r}^{(1)},\ f_r(x) = f_r^{(1)}(x),\ \phi_r(x) = \phi_r^{(1)}(x).$$

Exactly as in the proof of (2.4) we find that

$$\phi_r^{(p)}(x) = \frac{(r+p)!}{(r+1)!}\sum_{m=0}^{r-p}\frac{1}{(m+1)_p}\binom{r}{m}\binom{r-p}{m}x^m \quad (p \le r) \tag{4.4}$$

and

$$\phi_r^{(p)}(x) = \frac{(r+p)!}{(r+1)!}\sum_{m=0}^{r}\frac{(-1)^m}{(m+1)_p}\binom{r}{m}\binom{p-r+m-1}{m}x^m \quad (p > r). \tag{4.5}$$

In hypergeometric notation, both (4.4) and (4.5) become

$$\phi_r^{(p)}(x) = \frac{(r+p)!}{(r+1)!p!}F[-r+p,\ -r;\ p+1;\ x]. \tag{4.6}$$

Note that $\phi_r^{(p)}(x)$ is of degree $r-p$ for $p \le r$ and of degree r for $p > r$.

Since [1, p. 254, Eq. (2)]

$$P_n^{(p,p)}(x) = \frac{(p+1)_n}{n!}\left(\frac{x+1}{2}\right)^n F\left[-n,\ -n-p;\ p+1;\ \frac{x-1}{x+1}\right],$$

it follows that

$$\phi_{r+p}^{(p,p)}(x) = \frac{r!}{(p+1)_r}(1-x)^r P_r^{(p,p)}\left(\frac{1+x}{1-x}\right). \tag{4.7}$$

SECTION 5

We shall now obtain a generating function for $S_{n,r}^{(p)}$ in the following way. We have

$$\sum_{r=0}^{\infty} z^r \sum_{m=2r}^{\infty}\ \sum_{n=2r+p}^{\infty}\binom{m-r}{r}\binom{n-r-p}{r}x^m y^n = \sum_{r=0}^{\infty}\frac{x^{2r}y^{2r+p}z^p}{(1-x)^{-r-1}(1-y)^{-r-1}}$$
$$= y^p((1-x)(1-y) - x^2y^2z)^{-1}$$
$$= y^p(1 + xy - x^2y^2z - (x+y))^{-1}.$$

Replacing x by xy^{-1}, we have

$$\sum_{r=0}^{\infty} z^r \sum_{m=2r}^{\infty}\ \sum_{n=2r+p}^{\infty}\binom{m-r}{r}\binom{n-r-p}{r}x^m y^{n-m} = y^p(1 + x - x^2z - (xy^{-1} + y))^{-1}$$

$$= y^p(1 + x - x^2z)^{-1}\sum_{k=0}^{\infty}\frac{(xy^{-1}+y)^k}{(1+x-x^2z)^k}$$

$$= (1 + x - x^2z)^{-1}\sum_{j,k=0}^{\infty}\binom{j+k}{k}\frac{x^j y^{k-j+p}}{(1+x-x^2z)^{j+k}}.$$

Since we want only the terms on the right that are free of y, we take $j = k + p$. Thus

$$\sum_{r=0}^{\infty} z^r \sum_{n=2r+p}^{\infty}\binom{n-r}{r}\binom{n-r-p}{r}x^n = (1+x-x^2z)^{-p-1}\sum_{k=0}^{\infty}\binom{2k+p}{k}\frac{x^{k+p}}{(1+x-x^2z)^{2k}}.$$

Since [1, p. 70, Ex. 10]

$$\sum_{k=0}^{\infty}\binom{2k+p}{k}z^k = F\left[\frac{p+1}{2},\ \frac{p+2}{2};\ p+1;\ 4z\right] = (1-4z)^{-1/2}\left\{\frac{2}{1+(1-4z)^{1/2}}\right\}^p,$$

it follows easily that

$$\sum_{r=0}^{\infty} z^r \sum_{n=2r+p}^{\infty} (r+1)S_{n,r}^{(p)} x^{n-2r} = \frac{1}{R}\left(\frac{1+x-z-R}{2}\right)^p \quad (p \geq 0), \tag{5.1}$$

where

$$R = R(x, z) = ((1 - x^2 - 2(1+x)z + z^2)^{1/2}. \tag{5.2}$$

Since

$$\frac{\partial R}{\partial z} = \frac{z - 1 - x}{R},$$

it is easily verified that

$$\frac{\partial}{\partial z}\left(\frac{1+x-z-R}{2}\right)^p = \frac{p}{R}\left(\frac{1+x-z-R}{2}\right)^p.$$

Hence (5.1) yields

$$\frac{x^p}{p} + \sum_{r=0}^{\infty} z^{r+1} \sum_{n=2r+p}^{\infty} S_{n,r}^{(p)} x^{n-2r} = \frac{1}{p}\left(\frac{1+x-z-R}{2}\right)^p \quad (p > 0). \tag{5.3}$$

In the next place, by (4.2) and (4.3),

$$\sum_{n=2r+p} S_{n,r}^{(p)} x^{n-2r} = x^{-2r} f_r^{(p)}(x) = \frac{x^p}{(1-x)^{2r+1}} \phi_r^{(p)}(x).$$

Thus (5.3) becomes

$$1 + p\sum_{r=0} \frac{z^{r+1}}{(1-x)^{2r+1}} \phi_r^{(p)}(x) = \left(\frac{1+x-z-R}{2x}\right)^p.$$

Replacing z by $(1-x)^2 z$, we get

$$1 + p(1-x)\sum_{r=0} \phi_r^{(p)}(x) z^{r+1} = \left(\frac{1 + x - (1-x)^2 z - (1-x)R_0}{2x}\right)^p \quad (p > 0), \tag{5.4}$$

where

$$R_0 = (1 - 2(1+x)z - (1-x)^2 z^2)^{1/2}. \tag{5.5}$$

For $p = 1$, (5.4) reduces to

$$1 + (1-x)\sum_{r=0}^{\infty} \phi_r(x) z^{r+1} = \frac{1 + x - (1-x)^2 z - (1-x)R_0}{2x}. \tag{5.6}$$

It is easily verified that (5.6) is in agreement with (3.3).

Returning to (5.1), we have

$$\sum_{r=0}^{\infty} (r+1)\phi^{(p)}(x)\frac{z^r}{(1-x)^{2r+1}} = \frac{1}{R}\left(\frac{1+x-z-R}{2x}\right)^p,$$

so that

$$\sum_{r=0}^{\infty} (r+1)\phi_r^{(p)}(x) z^r = \frac{1}{R_0}\left(\frac{1 + x - (1-x)^2 z - (1-x)R_0}{2x}\right)^p \quad (p \geq 0). \tag{5.7}$$

Note that (5.7) holds for $p \geq 0$.

SECTION 6

As an immediate consequence of (5.4), we have

$$\left\{1 + p(1 - x)\sum_{s=0}^{\infty} \phi_s^{(p)}(x)z^{s+1}\right\}\left\{1 + q(1 - x)\sum_{t=0}^{\infty} \phi_t^{(q)}(x)z^{t+1}\right\}$$

$$= 1 + (p + q)(1 - x)\sum_{r=0}^{\infty} \phi_r^{(p+q)}(x)z^{r+1}.$$

Comparison of coefficients of z^{r+1} yields the convolution formula

$$\begin{aligned}(p + q)\phi_r^{(p+q)}(x) &= p\phi_r^{(p)}(x) + q\phi_r^{(q)}(x)\\ &\quad + pq(1 - x)\sum_{s=0}^{r-1} \phi_s^{(p)}(x)\phi_{r-s-1}^{(q)}(x) \qquad (p > 0,\ q > 0).\end{aligned} \tag{6.1}$$

Similarly, by (5.4) and (5.7),

$$\begin{aligned}(r + 1)\phi_r^{(p+q)}(x) &= (r + 1)\phi_r^{(q)}(x)\\ &\quad + p(1 - x)\sum_{s=0}^{r-1} (r - s)\phi_s^{(p)}(x)\phi_{r-s-1}^{(q)}(x) \qquad (p > 0,\ q \ge 0).\end{aligned} \tag{6.2}$$

In the next place, it is evident from (5.4) and (5.6) that

$$1 + p(1 - x)\sum_{r=0}^{\infty} \phi_r^{(p)}(x)z^{r+1} = \left\{1 + (1 - x)\sum_{r=0}^{\infty} \phi_r(x)z^{r+1}\right\}^p \qquad (p > 0). \tag{6.3}$$

For $p = q = 1$, (6.1) reduces to

$$2\phi_r^{(2)}(x) = 2\phi_r(x) + (1 - x)\sum_{s=0}^{r-1} \phi_s(x)\phi_{r-s-1}(x).$$

However, by (3.1), we have

$$\phi_r(x) = (1 - x)\phi_{r-1}(x) + x\sum_{s=0}^{r-1} \phi_s(x)\phi_{r-s-1}(x).$$

It follows that

$$2x\phi_r^{(2)}(x) = (1 + x)\phi_r(x) - (1 - x)^2\phi_{r-1}(x) \qquad (r > 0). \tag{6.4}$$

This formula can be generalized by means of the easily proved identity

$$2\binom{m}{r}\binom{m-p-1}{r} = \binom{m}{r}\binom{m-p}{r} + \binom{m-1}{r}\binom{m-p-1}{r} - \frac{r+p}{r}\binom{m-1}{r-1}\binom{m-p-1}{r-1}. \tag{6.5}$$

Multiplying both sides of (6.5) by x^m and summing over m, we get

$$2(r + 1)f_r^{(p+1)}(x) = (r + 1)(1 + x)f_r^{(p)}(x) - (r + p)x^2 f_{r-1}^{(p)}(x)$$

and therefore

$$2(r + 1)x\phi_r^{(p+1)}(x) = (r + 1)(1 + x)\phi_r^{(p)}(x) - (r + p)(1 - x)^2\phi_{r-1}^{(p)}(x). \tag{6.6}$$

For example, for $p = 2$, we get

$$\begin{aligned}4(r + 1)x^2\phi_r^{(3)}(x) &= (r + 1)(1 + x)^2\phi_r(x) - (2r + 3)(1 + x)(1 - x)^2\phi_{r-1}(x)\\ &\quad + (r + 2)(1 - x)^4\phi_{r-2}(x) \qquad (r > 1).\end{aligned} \tag{6.7}$$

Repeated application of (6.6) leads to a result of the form

$$(2x)^p\psi_r^{(p+1)}(x) = \sum_{s=0}^{p}(-1)^s c(p, r, s)(1+x)^{p-s}(1-x)^{2s}\psi_{r-s}(x) \quad (r \geq 0). \tag{6.8}$$

where

$$\psi_r^{(p)}(x) = (r+1)!\phi_r^{(p)}(x), \quad \psi_r(x) = (r+1)!\phi_r(x)$$

and the coefficients $c(p, r, s)$ are independent of x.

SECTION 7

We shall now state the binomial identities implied by (6.1) and (6.2). In terms of $f_r^{(p)}(x)$, (6.1) and (6.2) become

$$\begin{aligned}(p+q)f_r^{(p+q)}(x) &= px^q f_r^{(p)}(x) + qx^p f_r^{(q)}(x)\\ &\quad + pqx^2\sum_{s=0}^{r-1} f_s^{(p)}(x)f_{r-s-1}^{(q)}(x) \quad (p > 0,\ q > 0)\end{aligned} \tag{7.1}$$

and

$$\begin{aligned}(r+1)f_r^{(p+q)}(x) &= (r+1)f_r^{(q)}(x)\\ &\quad + p\sum_{s=0}^{r-1}(r-s)f_s^{(p)}(x)f_{r-s-1}^{(q)}(x) \quad (p > 0,\ q \geq 0),\end{aligned} \tag{7.2}$$

respectively. Using (4.2) and equating coefficients of x^n, we obtain the following identities.

$$\begin{aligned}(p+q)S_{n,r}^{(p+q)} &= pS_{n-q,r}^{(p)} + qS_{n-p,r}^{(p)}\\ &\quad + pq\sum_{j=0}^{n-2}\sum_{s=0}^{r-1} S_{j,s}^{(p)}S_{n-j-2,r-s-1}^{(q)} \quad (p > 0,\ q > 0)\end{aligned} \tag{7.3}$$

and

$$\begin{aligned}(r+1)S^{(p+q)} &= (r+1)S^{(q)}\\ &\quad + p\sum_{j=0}^{n-2}\sum_{s=0}^{r-1}(r-s)S_{j,s}^{(p)}S_{n-j-2,r-s-1}^{(q)} \quad (p > 0,\ q \geq 0).\end{aligned} \tag{7.4}$$

In particular, since

$$(r+1)S^{(0)} = \binom{n-r}{r}^2,$$

it is evident that, for $q = 0$, (7.4) reduces to

$$(r+1)S_{n,r}^{(p)} = \binom{n-r}{r}^2 + p\sum_{j=0}^{n-2}\sum_{s=0}^{r-1}\binom{j-s}{s}^2 S_{n-j-2,r-s-1}^{(p)} \quad (p > 0).$$

The special case, $p = 1$, was stated in the Introduction.

A second pair of identities is also implied by (6.1) and (6.2). Put

$$T_{r,m}^{(p)} = \frac{(r+p)!}{(r+1)!(m+1)_p}\binom{r}{m}\binom{r-p}{m} = \frac{1}{r+1}\binom{r+p}{m+p}\binom{r-p}{m}. \tag{7.5}$$

Then by (4.4) we have

$$\phi_r^{(p)}(x) = \sum_{m=0}^{r} T_{r,m}^{(p)}x^m \quad (p \geq 0). \tag{7.6}$$

Note that, by (4.4) and (4.5), (7.6) holds for all nonnegative p. Substituting from (7.6) in (6.1) and (6.2) and evaluating coefficients of x^m, we obtain the following two identities.

$$\begin{aligned}(p+q)T_{r,m}^{(p+q)} = {} & pT_{r,m}^{(p)} + qT_{r,m}^{(q)} + pq\sum_{s=0}^{r-1}\sum_{j=0}^{m} T_{s,j}^{(p)}T_{r-s-1,m-j}^{(q)} \\ & - pq\sum_{s=0}^{r-1}\sum_{j=0}^{m-1} T_{s,j}^{(p)}T_{r-s-1,m-j-1}^{(q)} \quad (p > 0,\ q > 0),\end{aligned} \tag{7.7}$$

$$\begin{aligned}(r+1)T_{r,m}^{(p+q)} = {} & (r+1)T_{r,m}^{(q)} + p\sum_{s=0}^{r-1}\sum_{j=0}^{m}(r-s)T_{s,j}^{(p)}T_{r-s-1,m-j}^{(q)} \\ & - p\sum_{s=0}^{r-1}\sum_{j=0}^{m-1}(r-s)T_{s,j}^{(p)}T_{r-s-1,m-j-1}^{(q)} \quad (p > 0).\end{aligned} \tag{7.8}$$

In particular, for $q = 0$, (7.8) reduces to

$$\begin{aligned}(r+1)T_{r,m}^{(p)} = {} & \binom{r}{m}^2 + p\sum_{s=0}^{r-1}\sum_{j=0}^{m}\binom{s}{j}^2 T_{r-s-1,m-j}^{(p)} \\ & - p\sum_{s=0}^{r-1}\sum_{j=0}^{m-1}\binom{s}{j}^2 T_{r-s-1,m-j-1}^{(p)} \quad (p > 0).\end{aligned}$$

We remark that (6.1) is implied by (6.2). To see this, multiply both sides of (6.2) by q, interchange p and q, and then add corresponding sides of the two equations. Similarly, it can be verified that (7.3) is implied by (7.4) and (7.7) is implied by (7.8).

REFERENCE

1. E. D. Rainville. *Special Functions*. New York: Macmillan, 1960.

*****

SOME EXTREMAL PROBLEMS ON DIVISIBILITY PROPERTIES OF SEQUENCES OF INTEGERS

PAUL ERDÖS

University of California, Los Angeles CA 90024

Dedicated to the memory of my friend Vern Hoggatt

A sequence of integers $A = \{a_1 < a_2 < \cdots < a_k \le n\}$ is said to have property $P_r(n)$ if no a_i divides the product of r other a's. Property $P(n)$ means that no a_i divides the product of the other a's. A sequence has property $Q(n)$ if the products a_ia_j are all distinct.

Many decades ago I proved the following theorems [2]:

Let A have property P_1 (i.e., no a_i divides any other). Then

$$\max k = \left[\frac{n+1}{2}\right].$$

The proof is easy.