FIBONACCI INDUCED GROUPS AND THEIR HIERARCHIES

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Not only was the late Professor Hoggatt a dedicated teacher and a prolific scholar, he was also a fine colleague and a reliable co-worker. In the summer of 1980 we met often, both of us being involved in Santa Clara's Undergraduate Research Participation program in mathematics. I showed him a rough draft of the few ideas expressed in this paper, and, while he was encouraging me to prepare it for The Fibonacci Quarterly, at the same time he asked me urgently to make it readable for a great variety of readers, "our readers" as he called them not without affection. In trying to comply with his request, I discovered that the paper became more than the communication of some results; it began to tell the story of how they were obtained, from simple well-known beginnings, through some redundant complications, toward a simple ending. May the effort be a stone in the monument to the memory of Dr. Verner E. Hoggatt, Jr.

1. INTRODUCTION

Sequences of integers give rise to algebraic structures and sequence hierarchies in various ways, some trivially, others by more sophisticated methods. A glance at a trivial example may help to grasp readily the subject matter of this paper.

Let N be the set of positive integers, Z the set of all integers. The function $s\colon N \to Z$, somehow defined, constitutes the sequence $s_1, s_2, s_3, \ldots$, where the arguments are written as subscripts. For every $z \in Z$, a function $t_z\colon N \to Z$ can be defined by $t_z(n) = zs_n$, thus constituting the sequence

$$t_{z1} = zs_1,\ t_{z2} = zs_2,\ t_{z3} = zs_3,\ \ldots .$$

Let $Z_t = \{t_z\colon z \in Z\}$. On Z_t one defines addition and multiplication by

$$t_a + t_b = t_{a+b} \quad \text{and} \quad t_a t_b = t_{ab}.$$

These definitions are not arbitrary or ad hoc; they amount to the usual pointwise addition and multiplication of functions, e.g.,

$$\begin{aligned}(t_a + t_b)(n) &= t_a(n) + t_b(n) = t_{an} + t_{bn} = as_n + bs_n\\ &= (a + b)s_n = t_{(a+b)n} = t_{a+b}(n).\end{aligned}$$

The result is an algebraic structure

$$(Z_t,\ +,\ \cdot,\ t_0,\ t_1)$$

where t_0 and t_1 are additive and multiplicative identities, respectively; t_0 is the sequence with all terms 0 and t_1 is s. This algebraic structure is clearly isomorphic to $(Z, +, \cdot, 0, 1)$, the familiar integral domain of the integers, by $\phi\colon Z \to Z_t$ with $\phi(z) = t_z$, thus being itself an integral domain.

The integral domain Z_t is a trivial example of an algebraic structure induced by the sequence s. As to the hierarchy involved, let the function $S\colon N \to Z_t$ be defined by $S(n) = t_{s(n)}$. The result is a sequence S with

$$S_1 = t_{s(1)},\ S_2 = t_{s(2)},\ S_3 = t_{s(3)},\ \ldots,$$

a sequence of sequences, such that each term of S is the element of Z_t that has the corresponding term of s as index; as a sequence, S is completely patterned after s. One could call S the second level of a hierarchy of which s is the first and lowest level. Starting with the sequence S one arrives in a similar way at the third level. For every $y, z \in Z$, let yt_z be the function $yt_z\colon N \to Z_t$

with $(yt_z)(n) = t_{yz}(n)$. For every $y \in Z$, let $T_y: N \to Z_t$ with $T_y(n) = yS_n$, $(= yt_{s(n)} = t_{ys(n)})$. Further, let $Z_T = \{T_y: y \in Z\}$. On Z_T one defines addition and multiplication pointwise, thus obtaining Z_T as an integral domain. The function $\mathcal{S}: N \to Z_T$ with $\mathcal{S}(n) = T_{s(n)}$ constitutes the sequence

$$\mathcal{S}_1 = T_{s(1)},\ \mathcal{S}_2 = T_{s(2)},\ \ldots,$$

the third level in the hierarchy. This construction can be repeated indefinitely.

As a concrete example of the above one could take any sequence of integers, but in this context one can as well take the Fibonacci sequence $f: N \to Z$ with $f_1 = 1$, $f_2 = 1$, and, for $n > 2$, $f_n = f_{n-1} + f_{n-2}$, yielding the well-known sequence

$$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 3,\ f_5 = 5,\ f_6 = 8,\ \ldots .$$

For every $z \in Z$, a function $g_z: N \to Z$ can be defined by $g_z(n) = zf_n$, thus constituting the sequence

$$g_{z1} = zf_1,\ g_{z2} = zf_2,\ g_{z3} = zf_3,\ \ldots .$$

Let $Z_g = \{g_z: z \in Z\}$. On Z_g one defines $g_a + g_b = g_{a+b}$ and $g_a g_b = g_{ab}$. The result is the integral domain $(Z_g, +, \cdot, g_0, g_1)$, isomorphic with the integral domain of the integers, induced by the sequence f. Let $F: N \to Z_g$ be defined by $F(n) = g_{f(n)}$. This yields the sequence F with terms

$$F_1 = g_1,\ F_2 = g_1,\ F_3 = g_2,\ F_4 = g_3,\ F_5 = g_5,\ F_6 = g_8,\ \ldots .$$

It should be noticed that again, for $n > 2$, $F_n = F_{n-1} + F_{n-2}$. The sequence F could be called a Fibonacci sequence of Fibonacci sequences, the second level of a hierarchy of which f is the first and lowest level. Continuing, for every $y, z \in Z$, let yg_z be the function $yg_z: N \to Z_g$ with $(yg_z)(n) = g_{yz}(n)$, $(= yzf_n)$. For every $z \in Z$, let G_z be the function $G_z: N \to Z_g$ with $G(n) = zF_n$, $(= zg_{f(n)} = g_{zf(n)})$. Let $Z_G = \{G_z: z \in Z\}$. Again introducing pointwise addition and multiplication on Z_G, one obtains Z_G as an integral domain. Let $\mathfrak{F}: N \to Z_G$ with $\mathfrak{F}(n) = G_{f(n)}$, then $\mathfrak{F}$ constitutes the sequence

$$\mathfrak{F}_1 = G_1,\ \mathfrak{F}_2 = G_1,\ \mathfrak{F}_3 = G_2,\ \mathfrak{F}_4 = G_3,\ \mathfrak{F}_5 = G_5,\ \mathfrak{F}_6 = G_8,\ \ldots,$$

the third level of the infinite hierarchy. The sequence is a Fibonacci sequence of Fibonacci sequences of Fibonacci sequences.

2. GENERATION OF A HIERARCHY OF GROUPS

One way of generalizing the Fibonacci sequence consists in extending its domain from N to Z. In this section let f denote the function $f: Z \to Z$ defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-2} + f_{n-1}$, or, $f_{n-2} = f_n - f_{n-1}$, where the arguments are again written as subscripts. Clearly, $f|N$, the restriction of f to N, yields the original Fibonacci sequence. The set of values $\{f_n: n \in Z\}$ can be pictured as an extension to the left of the original sequence:

$$\ldots,\ f_{-5} = 5,\ f_{-4} = -3,\ f_{-3} = 2,\ f_{-2} = -1,\ f_{-1} = 1,$$
$$f_0 = 0,\ f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 3,\ f_5 = 5,\ \ldots .$$

The restriction $f|N$ is nearly injective, spoiled only at the very beginning by $f_1 = f_2$. The extended f is only half as nice due to the identities formulated in Lemma 1.

Lemma 1: If $n \in N \cup \{0\}$ is even, then $f_{-n} = -f_n$,
if $n \in N$ is odd, then $f_n = f_{-n}$.

Proof:

Base step.—Trivially $f_{-0} = f_0 = 0 = -f_0$ and obviously $f_1 = 1 = f_{-1}$.

Induction step.—For $m > 1$, let it be assumed that the lemma holds for all $k \in N$ such that $k < m$. By definition,

$$f_{-m} = f_{-m+2} - f_{-m+1} = f_{-(m-2)} - f_{-(m-1)}.$$

If m is even, then $m - 2$ is even and $m - 1$ is odd, and hence, by the induction hypothesis,

$$f_{-m} = -f_{m-2} - f_{m-1} = -(f_{m-2} + f_{m-1}) = -f_m.$$

If m is odd, then $m - 2$ is odd, $m - 1$ is even, and the induction hypothesis yields

$$f_{-m} = f_{m-2} - (-f_{m-1}) = f_{m-2} + f_{m-1} = f_m.$$

The next lemma is an extension of a well-known lemma. The proof is extended to all the integers.

Lemma 2: For every $n \in Z$, let D_n be the determinant of the matrix

$$\begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}$$

If n is even, then $D_n = 1$; if n is odd, then $D_n = -1$.

Proof: Obviously $D_{-1} = -1$, $D_0 = 1$, and $D_1 = -1$. Moreover, for every $m \in Z$,

$$D_m = \begin{vmatrix} f_{m-1} & f_m \\ f_m & f_{m+1} \end{vmatrix} = \begin{vmatrix} f_{m-1} & f_{m-1} + f_{m-2} \\ f_m & f_m + f_{m-1} \end{vmatrix}$$

$$= 0 + \begin{vmatrix} f_{m-1} & f_{m-2} \\ f_m & f_{m-1} \end{vmatrix} = \begin{vmatrix} f_{m-3} + f_{m-2} & f_{m-2} \\ f_{m-2} + f_{m-1} & f_{m-1} \end{vmatrix}$$

$$= \begin{vmatrix} f_{m-3} & f_{m-2} \\ f_{m-2} & f_{m-1} \end{vmatrix} + 0 = D_{m-2}.$$

Hence, if n is even, $\frac{1}{2}|n|$ applications of the rule $D_m = D_{m-2}$, upward or downward, according to whether n is negative or positive, respectively, yield $D_n = D_0 = 1$. And if n is odd, $\frac{1}{2}(|n| - 1)$ applications of the rule, upward or downward, yield $D_n = D_{-1} = -1$ or $D_n = D_1 = -1$.

As is well known, the invertible 2 x 2 matrices with real entries have determinants $\neq 0$ and form a group under matrix multiplication with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as identity and $\begin{pmatrix} \frac{d}{D} & \frac{-b}{D} \\ \frac{-c}{D} & \frac{a}{D} \end{pmatrix}$ as the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where D is the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $\langle q \rangle$ be the cyclic subgroup generated by q.

Lemma 3: The cyclic group $\langle q \rangle$ is of infinite order and for every $n \in Z$,

$$q^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}.$$

Proof: (*a*) For $n \in N \cup \{0\}$ by straightforward induction.

$$q^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} f_{-1} & f_0 \\ f_0 & f_1 \end{pmatrix}.$$

Next, for $m \in N$, if $q^m = \begin{pmatrix} f_{m-1} & f_m \\ f_m & f_{m+1} \end{pmatrix}$, then

$$q^{m+1} = q^m q = \begin{pmatrix} f_{m-1} & f \\ f_m & f_{m+1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} f_m & f_{m-1} + f_m \\ f_{m+1} & f_m + f_{m+1} \end{pmatrix} = \begin{pmatrix} f_m & f_{m+1} \\ f_{m+1} & f_{m+2} \end{pmatrix}.$$

(*b*) For $n < 0$, let $n = -m$. Then $m \in N$ and by (a) above,

$$q^m = \begin{pmatrix} f_{m-1} & f_m \\ f_m & f_{m+1} \end{pmatrix}.$$

If m is even, then $D_m = 1$ and

$$q^n = q^{-m} = (q^m)^{-1} = \begin{pmatrix} f_{m+1} & -f_m \\ -f_m & f_{m-1} \end{pmatrix} = \begin{pmatrix} f_{-m-1} & f_{-m} \\ f_{-m} & f_{-m+1} \end{pmatrix} = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix},$$

by Lemmas 1 and 2. Similarly, if m is odd, then $D_m = -1$ and

$$q^n = q^{-m} = (q^m)^{-1} = \begin{pmatrix} -f_{m+1} & f_m \\ f_m & -f_{m-1} \end{pmatrix} = \begin{pmatrix} f_{-m-1} & f_{-m} \\ f_{-m} & f_{-m+1} \end{pmatrix} = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}.$$

(*c*) The infinite order of $\langle q \rangle$ is now obvious; for every $n > 0$, $q^n \neq q^0$, since $f_n \neq f_0$.

Since $\langle q \rangle$ is a cyclic group of infinite order, $q^m = q^n$ if and only if $m = n$. The function $s: Z \to \langle q \rangle$ with $s(n) = q^n$ is bijective. Moreover,

$$s(n + m) = q^{n+m} = q^n \cdot q^m = s(n) \cdot s(m).$$

Hence, the multiplicative group $\langle q \rangle$ is isomorphic to $(Z, +)$, the additive group of the integers. Since the elements of $\langle q \rangle$ are 2 x 2 matrices, they can be added by the usual matrix addition, but $\langle q \rangle$ is not closed under that addition. However, the following lemma holds.

Lemma 4: For every $n \in Z$, $q^n = q^{n-2} + q^{n-1}$, where + is the usual matrix addition.

Proof:

$$q^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix} = \begin{pmatrix} f_{n-3} + f_{n-2} & f_{n-2} + f_{n-1} \\ f_{n-2} + f_{n-1} & f_n + f_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} f_{n-3} & f_{n-2} \\ f_{n-2} & f_{n-1} \end{pmatrix} + \begin{pmatrix} f_{n-2} & f_{n-1} \\ f_{n-1} & f_n \end{pmatrix} = q^{n-2} + q^{n-1}.$$

The function s can be seen as a two-sided sequence with $s_n = q^n$ for every $n \in Z$, with $s_n \neq s_m$ for all $n,m \in Z$ such that $n \neq m$, and with $s_n = s_{n-2} + s_{n-1}$. Moreover, the elements of the sequence form an abelian group under multiplication.

Once s is established, one may as well dispense with the matrices. For every $n \in Z$, s_n is uniquely determined by the ordered triple (f_{n-1}, f_n, f_{n+1}), and conversely. Let T be the set of all ordered triples of consecutive members of the sequence $f: Z \to Z$ as defined above, ordered from left to right. Let $t: \langle q \rangle \to T$ with $t(q^n) = (f_{n-1}, f_n, f_{n+1})$. Let "multiplication" be defined on T by

$$(f_{m-1}, f_m, f_{m+1})(f_{n-1}, f_n, f_{n+1}) = (f_{m+n-1}, f_{m+n}, f_{m+n+1}).$$

Then

$$t(q^m q^n) = t(q^{m+n}) = (f_{m+n-1}, f_{m+n}, f_{m+n+1}) = (f_{m-1}, f_m, f_{m+1})(f_{n-1}, f_n, f_{n+1}) = t(q^m)t(q^n).$$

Thus, t establishes an isomorphism between $\langle q \rangle$ and T. Putting $F = ts$, the composition of s and t, one obtains $F: Z \to T$ with $F_n = (f_{n-1}, f_n, f_{n+1})$. Since both s and t are isomorphisms, so is F, and $(T, \cdot)$ is a multiplicative group isomorphic to the additive group of the integers. Moreover, using the familiar addition for ordered triples of numbers,

$$(a, b, c) + (a', b', c') = (a + a', b + b', c + c'),$$

one obtains, as in Lemma 4, $F_n = F_{n-2} + F_{n-1}$. Summarizing these results, one obtains the following lemma.

<u>*Lemma 5:*</u> Let $f: Z \to Z$ with $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-2} + f_{n-1}$. Let

$$T = \{(f_{n-1}, f_n, f_{n+1}): n \in Z\},$$

and let $F: Z \to T$ with $F_n = (f_{n-1}, f_n, f_{n+1})$. Let $F_n F_m = F_{n+m}$ and let

$$F_n + F_m = (f_{n-1} + f_{m-1}, f_n + f_m, f_{n+1} + f_{m+1}).$$

Then F is a bijective function constituting a two-sided sequence with terms $F_n, n \in Z$, and the property $F_n = F_{n-2} + F_{n-1}$. Moreover, the terms of F form an abelian group under multiplication, isomorphic to the additive group of the integers.

The group $(T, \cdot)$ may be called a Fibonacci induced group. The sequences f and F form the first and second levels of an infinite hierarchy. Next, one may consider the set of all ordered triples of consecutive members of the sequence $F: Z \to T$, ordered from left to right, say $\mathfrak{J} = \{(F_{n-1}, F_n, F_{n+1}): n \in Z\}$. Let $\mathfrak{F}: Z \to \mathfrak{J}$ with $\mathfrak{F}_n = (F_{n-1}, F_n, F_{n+1})$. Further, let $\mathfrak{F}_n \mathfrak{F}_m = \mathfrak{F}_{n+m}$ and

$$\mathfrak{F}_n + \mathfrak{F}_m = (F_{n-1} + F_{m-1}, F_n + F_m, F_{n+1} + F_{m+1}).$$

Although $\mathfrak{J}$ is not closed under addition, one still has $\mathfrak{F}_n = \mathfrak{F}_{n-2} + \mathfrak{F}_{n-1}$, because

$$\mathfrak{F}_n = (F_{n-1}, F_n, F_{n+1}) = (F_{n-3} + F_{n-2}, F_{n-2} + F_{n-1}, F_{n-1} + F_n) = (F_{n-3}, F_{n-2}, F_{n-1}) + (F_{n-2}, F_{n-1}, F_n) = \mathfrak{F}_{n-2} + \mathfrak{F}_{n-1}.$$

The terms of $\mathfrak{F}$ again form an abelian group under multiplication, isomorphic to the additive group of the integers. The identity element is $\mathfrak{F}_0 = (F_{-1}, F_0, F_1)$ and the inverse of $\mathfrak{F}_n$ is $\mathfrak{F}_{-n}$. Associativity and commutativity are inherited from the integers that serve as indices.

3. CONCLUSION: FROM TRIPLES TO q-TUPLES

The previous section resulted in groups of triples and their hierarchy. The group operation was induced by the multiplication of 2 × 2 matrices. Discarding the matrices, one can define this operation as well on ordered q-tuples ($q \in N$, $q > 1$) of consecutive terms of the extended, two-sided Fibonacci sequence, with the ordering from left to right.

The ordered pairs are the first to be considered. Again let $f: Z \to Z$ with $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-2} + f_{n-1}$. Let $P = \{(f_{n-1}, f_n): n \in Z\}$, the set of all ordered pairs of consecutive terms of f, ordered from left to right. Let $F^{(2)}$ be the function $F^{(2)}: Z \to P$ with $F_n^{(2)} = (f_{n-1}, f_n)$.

Lemma 6: The function $F^{(2)}$ is bijective.

Proof: The set $\{F_n^{(2)}: n \in Z\}$ is partitioned into the sets

$$A = \{F_n^{(2)}: n < 0\},$$

$$B = \{F_0^{(2)}, F_1^{(2)}\} = \{(1, 0), (0, 1)\},$$

and

$$C = \{F_n^{(2)}: n > 1\}.$$

The three sets are disjoint, because: (i) every pair in A contains a negative number (Lemma 1), no pair in B or C contains a negative number; (ii) every pair in B contains 0, no pair in C contains 0. Moreover, if $m \neq n$, then $F_m^{(2)} \neq F_n^{(2)}$; in B, trivially; in C, because the second coordinates of the pairs form the set $\{f_n : n > 1\}$ and for $n > 1$ the Fibonacci numbers are all different; in A, because the absolute values of the first coordinates of the pairs form the set $\{f_n: n > 1\}$, and hence are all different. Thus $F^{(2)}$ is injective. Obviously, $F^{(2)}$ is also surjective, and therefore bijective.

On $\{F_n^{(2)}: n \in Z\}$, let multiplication be defined by $F_m^{(2)}F_n^{(2)} = F_{m+n}^{(2)}$ and let addition be the usual addition of ordered pairs,

$$F_m^{(2)} + F_n^{(2)} = (f_{m-1} + f_{n-1}, f_m + f_n).$$

Clearly, the set is closed under multiplication but not under addition. However, $F_n^{(2)} = F_{n-2}^{(2)} + F_{n-1}^{(2)}$ because

$$F_n^{(2)} = (f_{n-1}, f_n) = (f_{n-3} + f_{n-2}, f_{n-2} + f_{n-1}) = (f_{n-3}, f_{n-2}) + (f_{n-2}, f_{n-1})$$
$$= F_{n-2}^{(2)} + F_{n-1}^{(2)}.$$

The terms of $F^{(2)}$ form an abelian group under multiplication, with

$$F_0^{(2)} = (f_{-1}, f_0) = (1, 0)$$

as identity element and $F_{-n}^{(2)}$ as the inverse of $F_n^{(2)}$. Associativity and commutativity are again inherited from the addition of the integers that serve as indices.

Passing to ordered q-tuples, let q be a fixed positive integer > 1. Let $Q = \{(f_{n-1}, f_n, \ldots, f_{n+q-2}): n \in Z\}$, the set of all ordered q-tuples of consecutive terms of f, ordered from left to right. Further, let $F^{(q)}: Z \to Q$ with $F_n^{(q)} = (f_{n-1}, \ldots, f_{n+q-2})$, (the choice of indices is for the sake of the previous ordered triples).

Lemma 7: For every $q \in N - \{1\}$, the function $F^{(q)}$ is bijective.

Proof: Obviously, $F^{(q)}$ is surjective. The proof that $F^{(q)}$ is injective is by straightforward induction on q.

Base step.—Given by Lemma 6.

Induction step.—Assume that the lemma holds for m, i.e., all ordered m-tuples of consecutive terms of f are different. Then clearly all ordered

$(m + 1)$-tuples are different also, since

$$(f_{n-1}, f_n, \ldots, f_{n+m-1}) = ((f_{n-1}, \ldots, f_{n+m-2}), f_{n+m-1})$$

and ordered pairs with different first coordinates are different.

On $\{F_n^{(q)} : n \in Z\}$, let multiplication be defined by $F_m^{(q)} F_n^{(q)} = F_{m+n}^{(q)}$ and let addition be the usual addition of ordered q-tuples

$$F_m^{(q)} + F_n^{(q)} = (f_{m-1} + f_{n-1}, \ldots, f_{m+q-2} + f_{n+q-2}).$$

Again, there is closure under multiplication, but not under addition. Still $F_n^{(q)} = F_{n-2}^{(q)} + F_{n-1}^{(q)}$ because

$$\begin{aligned} F_n^{(q)} &= (f_{n-1}, \ldots, f_{n+q-2}) = (f_{n-3} + f_{n-2}, \ldots, f_{n+q-4} + f_{n+q-3}) \\ &= (f_{n-3}, f_{n-2}, \ldots, f_{n+q-4}) + (f_{n-2}, f_{n-1}, \ldots, f_{n+q-3}) \\ &= F_{n-2}^{(q)} + F_{n-1}^{(q)}. \end{aligned}$$

The terms of $F^{(q)}$ form an abelian group under multiplication with

$$F_0^{(q)} = (f_{-1}, f_0, \ldots, f_{q-2})$$

as identity element and $F_{-n}^{(q)}$ as the inverse of $F_n^{(q)}$. Associativity and commutativity are again inherited from the integers. All this results in a generalization of Lemma 5.

Theorem 1: Let f: $Z \to Z$ with $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-2} + f_{n-1}$. For any fixed $q \in N - \{1\}$, let

$$Q = \{(f_{n-1}, f_n, \ldots, f_{n+q-2}) : n \in Z\}$$

and let $F^{(q)}$: $Z \to Q$ with

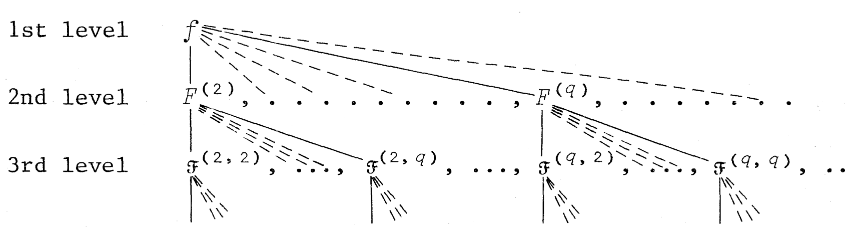
$$F_n^{(q)} = (f_{n-1}, f_n, \ldots, f_{n+q-2}).$$

Further, let $F_m F_n = F_{m+n}$ and

$$F_m + F_n = (f_{m-1} + f_{n-1}, f_m + f_n, \ldots, f_{m+q-2} + f_{n+q-2}).$$

Then $F^{(q)}$ is a bijective function constituting a two-sided sequence with terms $F_n^{(q)}$, $n \in Z$, and the property $F_n^{(q)} = F_{n-2}^{(q)} + F_{n-1}^{(q)}$. Moreover, the terms of $F^{(q)}$ form an abelian group under multiplication.

The hierarchy is now more complicated. Again calling f the first level, one obtains a second level which contains an infinity of sequences $F^{(q)}$, one for every $q \in N - \{1\}$. Each $F^{(q)}$ in its turn contributes infinitely many sequences $\mathfrak{F}^{(q,r)}$, $q, r \in N - \{1\}$, to the third level of the hierarchy, where the terms of $\mathfrak{F}^{(q,r)}$ consist of ordered r-tuples (from left to right) of consecutive terms of $F^{(q)}$. This construction can be repeated indefinitely. One can picture the hierarchy as a partial ordering, as follows:



One may avoid running out of letter types by noticing that the level number is one more than the number of coordinates in the tuples that form the superscripts. This way one can use capital letters for all levels, e.g., $F^{(q,r,s)}$

refers to a sequence of the fourth level, whose terms consist of ordered s-tuples of consecutive terms of $F^{(q,r)}$; this $F^{(q,r)}$ in its turn is a sequence of the third level, whose terms consist of ordered r-tuples of consecutive terms of $F^{(q)}$; and $F^{(q)}$ is a sequence of the second level whose terms consist of ordered q-tuples of consecutive terms of f.

The hierarchy results in a generalization of Theorem 1.

Theorem 2: Let $f: Z \to Z$ with $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-2} + f_{n-1}$. For every $m \in N$, let $q_1, \ldots, q_m \in N - \{1\}$. Let

$$Z^{(q_1)} = \{(f_{n-1}, f_n, \ldots, f_{n+q_1-2}): n \in Z\},$$

the set of all ordered q_1-tuples of consecutive terms of f. Let $F^{(q_1)}: Z \to Z^{(q_1)}$ with

$$F^{(q_1)} = (f_{n-1}, f_n, \ldots, f_{n+q_1-2}).$$

Further, let

$$Z^{(q_1, \ldots, q_m)} = \left\{\left(F_{n-1}^{(q_1, \ldots, q_{m-1})}, F_n^{(q_1, \ldots, q_{m-1})}, \ldots, F_{n+q_m-2}^{(q_1, \ldots, q_{m-1})}\right): n \in Z\right\},$$

the set of all ordered q_m-tuples of consecutive terms of $F^{(q_1, \ldots, q_{m-1})}$. Let

$$F_n^{(q_1, \ldots, q_m)}: Z \to Z^{(q_1, \ldots, q_m)}$$

with

$$F_n^{(q_1, \ldots, q_m)} = \left(F_{n-1}^{(q_1, \ldots, q_{m-1})}, F_n^{(q_1, \ldots, q_{m-1})}, \ldots, F_{n+q_m-2}^{(q_1, \ldots, q_{m-1})}\right).$$

Then $F^{(q_1, \ldots, q_m)}$ constitutes a two-sided sequence with terms $F_n^{(q_1, \ldots, q_m)}$, $n \in Z$, and the property

$$F_n^{(q_1, \ldots, q_m)} = F_{n-2}^{(q_1, \ldots, q_m)} + F_{n-1}^{(q_1, \ldots, q_m)}.$$

Moreover, the terms of $F^{(q_1, \ldots, q_m)}$ form an abelian group under the multiplication

$$F_n^{(q_1, \ldots, q_m)} F_p^{(q_1, \ldots, q_m)} = F_{n+p}^{(q_1, \ldots, q_m)}.$$

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EXPLORING AN ALGORITHM

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Dedicated to the memory of our dear friend and colleague, Vern

1. INTRODUCTION

We start with a simple algorithm for generating pairs L (left column) and R (right column) of Fibonacci numbers. In a slightly modified version we wish to investigate the ratios L/R as the number of iterations $n \to \infty$. This, it turns out, involves (ancient) history, geometry, number theory, linear algebra, numerical analysis, etc.!

2. THE BASIC ALGORITHMS

Let us consider a "computer project" (appropriate for the first assignment in an Introduction to Programming course):