HAIL TO THEE, BLITHE SPIRIT!

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It was in the mid-1940s that I left the Department of Applied Mathematics at Syracuse University in New York State to chair a small Department of Mathematics at the College of Puget Sound[1] in Tacoma, Washington. Among my first teaching assignments at the new location was a beginning class in college algebra and trigonometry. At the first meeting of this class I noticed, among the twenty-some assembled students, a bright-looking and somewhat roundish fellow who paid rapt attention to the introductory lecture.

As time passed I learned that the young fellow was named Verner Hoggatt, fresh from a hitch in the army and possessed of an unusual aptitude and appetite for mathematics. Right from the start there was little doubt in my mind that in Vern I had found the mathematics instructor's dream—a potential future mathematician. He so enjoyed discussing things mathematical that we soon came to devote late afternoons, and occasional evenings, to rambling around parts of Tacoma, whilst talking on mathematical matters. On these rambles I brought up things that I thought would particularly capture Vern's imagination and that were reasonably within his purview of mathematics at the time.

Since Vern seemed to possess a particular predilection and intuitive feeling for numbers and their beautiful properties, I started with the subject of Pythagorean triples, a topic that he found fascinating. I recall an evening, shortly after this initial discussion, when I thought I would test Vern's ability to apply newly acquired knowledge. I had been reading through Volume I of Jakob Bernoulli's *Opera* of 1744, and had come upon the alluring little problem: "Titius gave his friend, Sempronius, a triangular field of which the sides, in perticas, were 50, 50, and 80, in exchange for a field of which the sides were 50, 50, and 60. I call this a fair exchange." I proposed to Vern that, in view of the origin of this problem, we call two noncongruent isosceles triangles a *pair of Bernoullian triangles* if the two triangles have integral sides, common legs, and common areas. I invited Vern to determine how we might obtain pairs of Bernoullian triangles. He immediately saw how such a pair can be obtained from any given Pythagorean triangle, by first putting together two copies of the Pythagorean triangle with their shorter legs coinciding and then with their longer legs coinciding. He pointed out that from his construction, the bases of such a Bernoullian pair are even, whence the common area is an integer, so that these Bernoullian triangles are Heronian.

On another ramble I mentioned the problem of cutting off in a corner of a room the largest possible area by a two-part folding screen. I had scarcely finished stating the problem when Vern came to a halt, his right arm at the same time coming up to a horizontal position, with an extended forefinger. "There's the answer," he said. I followed his pointing finger, and there, at the end of the block along which we were walking, was an octagonal stop sign.

There was a popular game at the time that was, for amusement, engaging many mathematicians across the country. It had originated in a problem in *The American Mathematical Monthly*. The game was to express each of the numbers from 1 through 100 in terms of precisely four 9s, along with accepted mathematical symbols of operation. For example

$$1 = 9/9 + 9 - 9 = 99/99 = (9/9)^{9/9},$$

[1]*Now the University of Puget Sound.*

$$2 = 9/9 + 9/9 = .\dot{9} + .\dot{9} + 9 - 9,$$
$$3 = \sqrt{\sqrt{9}\sqrt{9}} + 9 - 9 = (\sqrt{9}\sqrt{9}\sqrt{9})/9.$$

The next day Vern showed me his successful list. In this list were the expressions

$$67 = (.\dot{9} + .\dot{9})^{\sqrt{9}!} + \sqrt{9} = (\sqrt{9} + .\dot{9})^{\sqrt{9}} + \sqrt{9},$$
$$68 = \sqrt{9}\text{¡}(\sqrt{9}!\sqrt{9}! - \sqrt{9}\text{¡}),$$
$$70 = (9 - .\dot{9})9 - \sqrt{9}\text{¡} = (.\dot{9} + .\dot{9})^{\sqrt{9}!} + \sqrt{9}!,$$

where the inverted exclamation point, ¡, indicates subfactorial.[2] For all the other numbers from 1 through 100, Vern had been able to avoid both exponents and subfactorials, and so he now tried to do the same with 67, 68, and 70, this time coming up with

$$67 = \sqrt{9!/(9 \times 9) + 9},$$
$$68 = (\sqrt{9}!)!/9 - \sqrt{9}! - \sqrt{9}!,$$
$$70 = (9 + .\dot{9})(\sqrt{9}! + .\dot{9}).$$

It would take too much space to pursue further the many many things we discussed in our Tacoma rambles, but, before passing on to later events, I should point out Vern's delightful wit and sense of humor. I'll give only one example. The time arrived in class when I was to introduce the concept of mathematical induction. Among some preliminary examples, I gave the following. "Suppose there is a shelf of 100 books and we are told that if one of the books is red then the book just to its right is also red. We are allowed to peek through a a vertical slit, and discover that the sixth book from the left is red. What can we conclude?" Vern's hand shot up, and upon acknowledging him, he asked, "Are they all good books?" Not realizing the trap I was walking into, I agreed that we could regard all the books as good ones. "Then," replied Vern, "*all* the books are red." "Why?" I asked, somewhat startled. "Because all *good* books are read," he replied, with a twinkle in his eye.

It turned out that I stayed only the one academic year at the College of Puget Sound, for I received an attractive offer from Professor Milne of Oregon State College[3] to join his mathematics staff there. The hardest thing about the move was my leave-taking of Vern. We had a last ramble, and I left for Oregon.

I hadn't been at Oregon State very long when, to my great joy and pleasure, at the start of a school year I found Vern sitting in a couple of my classes. He had decided to follow me to Oregon. We soon inaugurated what became known as our "oscillatory rambles." Frequently, after our suppers, one of us would call at the home of the other (we lived across the town of Corvallis from one another), and we would set out for the home of the caller. Of course, by the time we reached that home, we were in the middle of an interesting mathematical discussion, and so returned to the other's home, only to find that a new topic had taken over which needed further time to conclude. In this way, until the close of a discussion happened to coincide with the reaching of one of our homes, or simply because of the lateness of the hour, we spent the evening in oscillation.

Our discussions now were more advanced than during our Tacoma rambles. I recall that one of our earliest discussions concerned what we called *well-defined*

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[2] $n\text{¡} = n![1 - 1/1! + 1/2! - 1/3! + \cdots + (-1)^n n!]$.

[3] Now Oregon State University.

Euclidean constructions. Suppose one considers a point of intersection of two loci as *ill-defined* if the two loci intersect at the point in an angle less than some given small angle θ, that a straight line is *ill-defined* if the distance between the two points that determine it is less than some given small distance d, and that a circle is *ill-defined* if its radius is less than d; otherwise, the construction will be said to be *well-defined*. We proved that *any Euclidean construction can be accomplished by a well-defined one*. This later constituted our first jointly published paper (in *The Mathematics Teacher*). Another paper (published in *The American Mathematical Monthly*) that arose in our rambles, and an expansion of which became Vern's master's thesis, concerned the derivation of hyperbolic trigonometry from the Poincaré model. We researched on many topics, such as Schick's theorem, nonrigid polyhedra, new matrix products, vector operations as matrices, a quantitative aspect of linear independence of vectors, trihedral curves, Rouquet curves, and a large number of other topics in the field of differential geometry.

We did not forego our former interest in recreational mathematics. The number game of the Tacoma days had now evolved into what seemed a much more difficult one, namely, to express the numbers from 1 through 100 by arithmetic expressions that involve each of the ten digits $0, 1, \ldots, 9$ once and only once. This game was completely and brilliantly solved when Vern discovered that, for any nonnegative integer n,

$$\log_{(0+1+2+3+4)/5}\left\{\log_{\sqrt{\sqrt{\cdots\sqrt{(-6+7+8)}}}} 9\right\} = n,$$

where there are n square roots in the second logarithmic base. Notice that the ten digits appear in their natural order, and that, by prefixing a minus sign if desired, Vern had shown that *any integer*, positive, zero, or negative, can be represented in the required fashion.[4]

A little event that proved very important in Vern's life took place during our Oregon association. When I was first invited to address the undergraduate mathematics club at Oregon State, I chanced to choose for my topic, "From rabbits to sunflowers," a talk on the famous Fibonacci sequence of numbers. Vern, of course, attended my address, and it reawakened in him his first great mathematical interest, the love of numbers and their endless fascinating properties. For weeks after the talk, Vern played assiduously with the beguiling Fibonacci numbers. The pursuit of these and associated numbers became, in time, Vern's major mathematical activity, and led to his eventual founding of *The Fibonacci Quarterly*, devoted chiefly to the study of such numbers. During his subsequent long and outstanding tenure at San Jose State University, Vern directed an enormous number of master's theses in this area, and put out an amazing number of attractive papers in the field, solo or jointly with one or another of his students. He became *the* authority on Fibonacci and related numbers.

After several years at Oregon State College, I returned east, but Vern continued to inundate me with copies of his beautiful findings. When I wrote my *Mathematical Circles Squared* (Prindle, Weber & Schmidt, 1972), I dedicated the volume

To VERNER E. HOGGATT, JR.
who, over the years, has sent me more
mathematical goodies than anyone else

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[4]Another entertaining number game that we played was that of expressing as many of the successive positive integers as possible in terms of not more than three π's, along with accepted symbols of operation.

The great geographical distance between us prevented us from seeing one another very often. I did, on my way to lecturing in Hawaii, stop off to see Vern, and I spent a few days with him a couple of years later when I lectured along the California coast. He once visited me at the University of Maine, when, representing his university, he came as a delegate to a national meeting of Phi Kappa Phi (an academic honorary that was founded at the University of Maine). For almost four decades I had the enormous pleasure of Vern's friendship, and bore the flattering title, generously bestowed upon me by him, of his "mathematical mentor."

In mathematics, Vern was a skylark, and I regret, far more than I can possibly express, the sad fact that we now no longer will hear further songs by him. But, oh, on the other hand, how privileged I have been; I heard the skylark when he first started to sing.

Hail to thee, blithe Spirit!
Bird thou never wert,
That from Heaven, or near it,
Pourest thy full heart
In profuse strains of unpremeditated art.

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DIAGONAL SUMS IN THE HARMONIC TRIANGLE

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Dedicated to the memory of my colleague and friend, Verner Hoggatt

Leibniz's harmonic triangle is related to reciprocals of the elements of Pascal's triangle, and was developed in summing infinite series by a telescoping process as discussed by Kneale [1] and Price [2], among others. Here, we find row sums and rising diagonal sums for the harmonic triangle.

1. PROPERTIES OF THE HARMONIC TRIANGLE

The harmonic triangle of Leibniz

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\begin{array}{ccccccccccccc}
\frac{1}{1} & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \frac{1}{6} & & \frac{1}{7} \;\cdots \\
& \frac{1}{2} & & \frac{1}{6} & & \frac{1}{12} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{42} \;\cdots & \\
& & \frac{1}{3} & & \frac{1}{12} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{105} \;\cdots & & \\
& & & \frac{1}{4} & & \frac{1}{20} & & \frac{1}{60} & & \frac{1}{140} \;\cdots & & & \\
& & & & \frac{1}{5} & & \frac{1}{30} & & \frac{1}{105} \;\cdots & & & & \\
& & & & & \frac{1}{6} & & \cdots & & \cdots & & &
\end{array}
$$

is formed by taking successive differences of terms of the harmonic series.