Had Vern Hoggatt been able to coauthor this article he would no doubt have found many more results. Perhaps our readers will celebrate his memory by looking for further results themselves.

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FRACTIONAL PARTS $(nr - s)$, ALMOST ARITHMETIC SEQUENCES, AND FIBONACCI NUMBERS

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To the memory of Vern Hoggatt, with gratitude and admiration.

Except where noted otherwise, sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are understood to satisfy the following requirements, as stated for $\{a_n\}$:

(i) the indexing set $\{n\}$ is the set of *all* integers;
(ii) a_n is an integer for every n;
(iii) $\{a_n\}$ is a strictly increasing sequence;
(iv) the least positive term of $\{a_n\}$ is a_1.

We call $\{a_n\}$ *almost arithmetic* if there exist real numbers u and B such that

$$|a_n - un| < B \tag{1}$$

for all n, and we write $a_n \sim un$ if (1) holds for some B and all n.

Suppose r is any irrational number and s is any real number. Put

$$c_m = [mr - s] = \text{the greatest integer less than or equal to } mr - s,$$

and let b be any nonzero integer. It is easy to check that $c_{m+b} - c_m = [br]$, if $(mr - s) < (-br)$, and $= [br] + 1$, otherwise.

Let a_n be the nth term of the sequence of all m satisfying $c_{m+b} - c_m = [br]$.

In the following examples, $r = (1 + \sqrt{5})/2$, the golden mean, and $s = 1/2$. Selected values of m and c_m are:

-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16
-9, -7, -6, -4, -3, -1, 1, 2, 4, 5, 7, 9, 10, 12, 14, 15, 17, 18, 20, 22, 23, 25.

When $b = 1$ we have $[br] = 1$, and selected values of n and a_n are:

-1, 0, 1, 2, 3, 4, 5, 6
-4, -2, 1, 3, 6, 9, 11, 14.

When $b = 2$ we have $[br] = 3$, and selected values of n and a_n are:

-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11
-5, -4, -3, -2, 0, $\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{5}$, 6, $\underline{8}$, 9, 10, 11, $\underline{13}$, 14.

Note here the presence of Fibonacci numbers among the a_n. Methods given in this note can be used to confirm that the Fibonacci sequence is a subsequence of $\{a_n\}$ in the present case.

When $b = -2$ we have $[br] = -4$, and selected values of n and a_n are:

$$\begin{matrix} 0, & 1, & 2, & 3, & 4 \\ -4, & 1, & 6, & 9, & 14 \end{matrix}$$

The main purpose of this note is to give an elementary and constructive method of proving, in general, that the *jump-sequence* $\{a_n\}$ is almost arithmetic. To accomplish this, we must solve the inequality $(mr - s) < (-br)$ for m. The method of solution, when applied to the case $r = (1 + \sqrt{5})/2$, leads to a number of identities involving Fibonacci numbers, Lucas numbers, and the greatest integer function.

Lemma 1: Suppose $a_n \sim un$, where $u > 1$. Let $\{a_n^*\}$ be the complement of $\{a_n\}$, that is, the sequence of integers not in $\{a_n\}$, indexed according to requirements i-iv. Then

$$a_n^* \sim \frac{u}{u - 1}n.$$

Lemma 2: Suppose $a_n \sim un$ and $b_n \sim vn$. Then the composite $c_n = b_{a_n}$ satisfies $c_n \sim uvn$.

Lemma 3: Suppose $a_n \sim un$ and $b_n \sim vn$, where $a_j \neq b_k$ for all j and k. Let $\{c_n\}$ be the union of $\{a_n\}$ and $\{b_n\}$. Then

$$c_n \sim \left(\frac{1}{u} + \frac{1}{v}\right)^{-1} n.$$

Proofs of the three lemmas found in [7] for positive n can be extended readily to the case of all integers n.

Theorem: Suppose r is an irrational number, s a real number, and b a nonzero integer. Let $\{a_n\}$ be the sequence of integers m satisfying $(mr - s) \leq (br)$. Then $a_n \sim n/(br)$.

Proof: First, we note that $mr - s$ can be an integer for at most one value of m, and that whether the sequence $\{a_n\}$ is almost arithmetic does not depend on whether it contains such an m. Accordingly, we shall assume that all fractional parts which occur in this proof are positive. Also without loss we assume that $0 < r < 1$.

Suppose $b \geq 1$, and let $p = [br]$. If

$$(mr - s) \leq (br), \tag{2}$$

then for $k = [mr - s]$, the integer m must lie in the interval

$$J_k = \left(\frac{k + s}{r}, \frac{k + s}{r} + b - \frac{p}{r}\right].$$

Conversely, any m in such an interval satisfies (2) with $k = [mr - s]$.

Now let $q = [(br)/r]$, the greatest integer i satisfying $i - b + \frac{p}{r} < 0$. Then

$$\left(\frac{k + s}{r}\right) \geq q - b + \frac{p}{r}$$

for all integers k, so that for $q \geq 1$ and $i = 1, 2, \ldots, q$, the integers

$$m_k = \left[\frac{k + s}{r}\right] + i, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{3}$$

satisfy (2). Each of these sequences $\{m_k\}$ is almost arithmetic with slope $1/r$.

If $q = 0$ then some interval J_k contains no integer. In this case the solutions of (2) are the integers $\left[\frac{k + s}{r}\right] + 1$ for which $\left(\frac{k + s}{r}\right) \geq 1 - b + \frac{p}{r}$. If $q \geq 1$, then the integers $\left[\frac{k + s}{r}\right] + q + 1$, where

(4) $$\left(\frac{k+s}{r}\right) \geq q + 1 - b + \frac{p}{r},$$

are solutions of (2), along with the solutions given in (3), and by definition of q there are no other solutions. The two inequalities involving $\left(\frac{k+s}{r}\right)$ are each equivalent to

(5) $$\left(\frac{k+s}{r}\right) \geq \left(\frac{p}{r}\right).$$

By Lemma 2 the sequences $\left[\frac{k+s}{r}\right] + 1$ and $\left[\frac{k+s}{r}\right] + q + 1$ are almost arithmetic if the sequence of k satisfying (5) is so. By Lemma 1 this is indeed the case if the complementary sequence, consisting of all integers k satisfying

$$\left(\frac{k+s}{r}\right) < \left(\frac{p}{r}\right)$$

is almost arithmetic. Except for at most one k, this inequality is equivalent to

(6) $$(kr' - s') \leq (pr'),$$

where $r' = (1/r)$ and $s' = -s/r$.

As (6) is of the same form as (2), we note that with a finite number of applications of the process from (2) to (6) the integer p decreases to 0, since initially $0 \leq p \leq b - 1$. When $p = 0$, the number on the right-hand side of (4) is 1, indicating that there are no further values of k to be found. By forming the union of the (pairwise disjoint) solution sequences which have been found, we get an almost arithmetic sequence, by Lemma 3.

Suppose now $b \leq -1$. Then the integers m satisfying $(-mr + s) \leq (-br)$ form an almost arithmetic sequence. Thus, by Lemma 1, those integers m satisfying $(-mr + s) > (-br) = 1 - (br)$, or equivalently, $(mr - s) = 1 - (-mr + s) < (br)$, form an almost arithmetic sequence.

We have finished proving that $\{a_n\}$ is almost arithmetic. It remains to see that the number u in (1) is $1/(br)$.

If $b = 1$, then the a_n are the numbers $\left[\frac{k+s}{r}\right] + 1$, $k = 0, \pm 1, \pm 2, \ldots,$ as already proved, and hence $a_n \sim n/(r)$. For an induction hypothesis, suppose, for $b \geq 2$, that for all $d \leq b - 1$ the sequence $\{c_n\}$ of solutions m of

(7) $$(mr' + s') \leq (dr')$$

satisfies $c_n \sim n/(dr')$, for any given positive irrational r' and real s'. Let $\{b_n\}$ be the sequence of solutions of (7) where $r' = (1/r)$, $s' = -s/r$, and $d = p = [br] \leq b - 1$.

Let $\{b_n^*\}$ be the complement of $\{b_n\}$, so that

$$b_n^* \sim \frac{n}{1 - (p(1/r))} = \frac{n}{1 - (p/r)},$$

by Lemma 2. There are no other solutions if $q = 0$, and if $q \geq 1$, the remaining solutions are simply

$$f_{k,i} = \left[\frac{k+s}{r}\right] + i, \qquad k = 0, \pm 1, \pm 2, \ldots; \; i = 1, 2, \ldots, q,$$

as already proved. Since $f_{k,i} \sim k/r$ for $i = 1, 2, \ldots, q$, we have, by Lemma 3, for $q \geq 1$:

$$a_n \sim \frac{n}{qr + r - r(p/r)} = \frac{n}{[(br)/r]r + r - r([br]/r)}$$

$$= \frac{n}{[(br)/r]r + r - r(-(br)/r)} = n/(br).$$

In case $q = 0$, we find similarly $a_n \sim \dfrac{n}{r - r(p/r)} = n/(br)$.

Finally, suppose $b \leq -1$. The integers m satisfying $(-mr + s) > (-br)$ form a sequence $\{c_n\}$ satisfying $c_n \sim n/(-br)$. For the complement $\{a_n\} = \{c_n^*\}$, we have $a_n \sim n/(br)$, by Lemma 1.

Corollary 1: Suppose r is an irrational number and s is a real number. Suppose a and b are nonzero integers such that $(ar) < (br)$. Let $\{a_n\}$ be the sequence of integers m satisfying $(ar) < (mr - s) \leq (br)$. Then

$$a_n \sim \frac{n}{(br) - (ar)}.$$

Proof: Let $\{f_n\}$ and $\{h_n\}$ be the solution sequences of the inequalities $(mr - s) \leq (ar)$ and $(mr - s) \leq (br)$, respectively. The sequence $\{a_n\}$ is, in the terminology of [7], the relative complement of $\{f_n\}$ in $\{h_n\}$. Applying the method used in [7], we conclude that

$$a_n \sim \frac{n}{(br) - (ar)}.$$

Fraenkel, Mushkin, and Tassa [3] have obtained results indicated in their title, "Determination of $[n\theta]$ by Its Sequence of Differences." The theorem in this present note supplements those results. We may ask, for example, for a sequence $\{c_n\}$ whose consecutive differences are all 1's and 2's, determined by the rule $c_{n+1} - c_n = 1$ for exactly those n of the form $[m\theta - \phi]$, where θ and ϕ are given. The question leads to the following corollary.

Corollary 2: Suppose θ is a positive irrational number and ϕ is a real number. Let $h = [\theta]$, and let $\{c_n\}$ be the sequence determined by $c_{n+1} - c_n = h$ for exactly those n of the form $[m\theta - \phi]$ and $= h + 1$ otherwise. Then

$$c_n = [n + nh - n/\theta - \phi/\theta], \qquad n = 0, \pm 1, \pm 2, \ldots .$$

Proof: We have $c_{n+1} - c_n = h$ for exactly those n satisfying $(nr - s) < (-r)$, where $r = 1 + h - 1/\theta$ and $s = \phi/\theta$. These n are the integers of the form $[m/(-r) - s/(-r)]$, but this is $[m\theta - \phi]$.

The method of proof of the theorem readily shows that for any irrational r and any real s, the sequences given by

$$\left[\frac{n + s}{(r)}\right] \quad \text{and} \quad \left[\frac{n - s}{(-r)}\right] \tag{8}$$

are complementary (except that one term, and only one, can be common to the two sequences, as when $n = s = 0$). This fact is a generalization of the well-known result by Beatty [1], obtained here by putting $s = 0$ and restricting the sequences to positive integers and r to the unit interval.

Corresponding to (8), the jump-sequence $\{a_n\}$ of indexes m such that $[mr + r - s] - [mr - s] = [r]$ is given by $m = \left[\dfrac{n + s}{(r)}\right] + 1$, and the complementary jumps of size $[r] + 1$ occur at m of the form $\left[\dfrac{n - s}{(r)}\right] + 1$.

Explicit results for $b = 2$ and only positive terms are also easy to state, in two cases: If $(r) < 1/2$, then the three sequences

$$\left[\frac{n + s}{(r)}\right], \left[\frac{n + s}{(r)}\right] + 1, \left[\frac{\left[\dfrac{n - s/(-r)}{(-1/(-r))}\right] + 1 - s}{(-r)}\right]$$

are complementary, and if $(r) > 1/2$, then the sequences

$$(9) \qquad \left[\frac{n-s}{(-r)}\right], \quad \left[\frac{n-s}{(-r)}\right]+1, \quad \left[\frac{\left[\frac{n+s/(r)}{(-1/(r))}\right]+1+s}{(r)}\right]$$

are complementary.

For $(r) < 1/2$, the jump-sequence of m such that $[mr + 2r - s] - [mr - s] = [2r]$ is given by the union of the sequences $\left[\frac{n-s}{(-r)}\right] + 1$ and $\left[\frac{n-s}{(-r)}\right] + 2$, and jumps of size $[2r] + 1$ occur at integers $m = \left[\frac{k+s}{(r)}\right] + 1$, where k has the form $\left[\frac{n - s/(-r)}{(-1/(r))}\right] + 1$.

It is of historical interest that Hecke [4] first proved the theorem of this note in the case $s = 0$. That (br) *must* equal (jr) for some integer j in order for $\{c_m\}$ to be an almost arithmetic sequence, for the case $s = 0$, was proved by Kesten [6].

Taking r in (9) to be $(1 + \sqrt{5})/2$ leads to a number of identities involving Fibonacci numbers. For example, with $s = 0$, the three sequences in (9) may be written

$$(9') \qquad \left[\frac{2n}{3-\sqrt{5}}\right], \quad \left[\frac{2n}{3-\sqrt{5}}\right]+1, \quad \left[\frac{2\left[\frac{2n}{\sqrt{5}-1}\right]+2}{\sqrt{5}-1}\right]$$

It is easy to prove that the first two of these sequences contain all the Fibonacci numbers. In fact, the method of Bergum [2] can be used to show that

$$F_{n+2} = \begin{cases} \left[\dfrac{2F_n}{3-\sqrt{5}}\right] & \text{for odd } n \geq 1 \\[2ex] \left[\dfrac{2F_n}{3-\sqrt{5}}\right] + 1 & \text{for even } n \geq 0 \end{cases}$$

and

$$\left[\frac{2\left[\dfrac{2F_n}{\sqrt{5}-1}\right]+2}{\sqrt{5}-1}\right] = \begin{cases} F_{n+3} + 1 & \text{for odd } n \geq 1 \\ F_{n+3} - 1 & \text{for even } n \geq 0. \end{cases}$$

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