HOMMAGE À ARCHIMÈDE

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Verner Hoggatt and I were friends and colleagues for twenty years. He was a person with special properties who studied mathematical objects with special properties. In addition to his incomparable knowledge of all things Fibonacci was his remarkable storehouse of appreciation for geometrical matters of all kinds. I'm glad that he very much liked my "1:2:3" result connected with Figure 1, below; he gave me several of the references which appear in the brief, annotated bibliography. Of course, he liked the fact that his old friend

$$x^2 - x - 1 = 0$$

had motivated my pleasant little discovery; and he, too, did not at all hesitate to show his students some simple things that opened up interesting, broader and longer, avenues for them to pursue in the literature and in their private studies. Those of you who are teachers are invited to show these few paragraphs to your students. In some ways, the best way to remember a friend is to try to emulate him. So, in that spirit, I offer this leisurely little essay.

In the central Quad at San Jose State University, across from the landmark Tower, there now stands a seven foot bronze abstract sculpture, *Hommage à Archimède*, which provides an already pleasant place with an additional pleasant intellectual sweep. This handsome bronze tribute to Archimedes, made possible by contributions from friends of the School of Science of the University, incorporates several noteworthy scientific and artistic design ideas which are dealt with below.

For nearly two decades I had entertained the hope of placing some abstract sculpture on campus that would involve the Archimedes-related design in Figure 1. This hope was known to Kathleen Cohen, Chairman of our Art Department, who introduced me to Robert J. Knight, a sculptor who was spending some time last year on our campus.

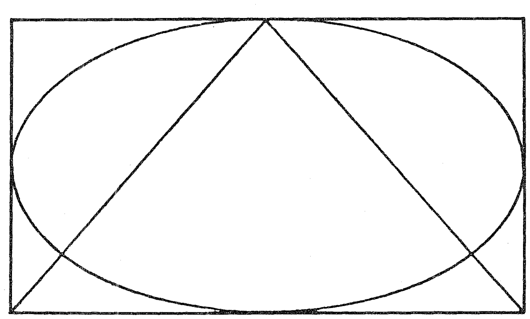

Fig. 1

Knowing that I did not want some heroic-bearded-Russian-heavy-General-type figure, the sculptor cooperatively modified one of his existing graceful abstract models to accommodate that Archimedean design, the only element of the whole piece which will make any immediate sense to a knowledgeable observer. Concerning the sculpture as a whole, I can only offer this quotation from a page in an Art Department brochure: "It is an exploration of the figurative formula and displacement of space as it relates to the human form."

There are many who think of Archimedes (287-212 B.C.) as the greatest intellect of antiquity. He is always listed among the top four or five mathematicians who have ever lived. In his time, and even much later, mathematicians were natural philosophers, natural scientists, if you please. Only in times much closer to our own do we find a greater scientific abstraction and consequent apparent separation of many mathematicians from other parts of the quest to understand nature. I do not know if Archimedes ever studied botany, say, but the existence of his significant interest and work in important areas of science other than pure mathematics is well documented. He wrote numerous masterpieces—on optics, hydrostatics, theoretical mechanics, astronomy, and mathematics, for example. It is true that, although he was surely a very good engineer, he did not regard very highly his own dramatic mechanical contrivances.

As indicated by his wishes regarding what should appear on his gravestone, Archimedes did most highly value a figure something like Figure 2, which refers to some beautiful geometry connected with his fundamental work as a primary and

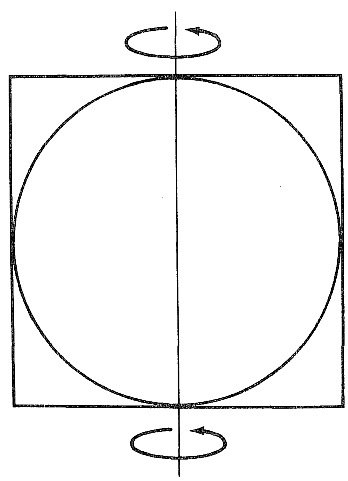

Fig. 2

impeccable forerunner of those who much later established modern integral calculus. (The Roman statesman Cicero wrote about and restored the Archimedes gravestone when it was rediscovered long after Archimedes had died.) That figure, a square with an inscribed circle, refers to this result of his: If the figure is rotated in space about that central vertical axis, the resulting sphere and cylinder have *volumes* which are as 2 is to 3; and their *surface areas* are *also* as 2 : 3. Archimedes, having discovered and appreciated much beautiful geometry, would certainly have understood what Edna St. Vincent Millay was saying (years later): "Euclid alone has looked on Beauty bare."

Democritus, who lived before Archimedes, knew the following result about Figure 3: The volumes of the cone and cylinder which are generated when the triangle and rectangle are rotated as indicated are as 1 : 3.

Over the years, some calculus students have been told to consider Figure 4 —which involves a square, not just *any* rectangle—and to use (modern) elementary calculus tools to find this beautiful result:

(α) Cone : Sphere : Cylinder = 1 : 2 : 3.

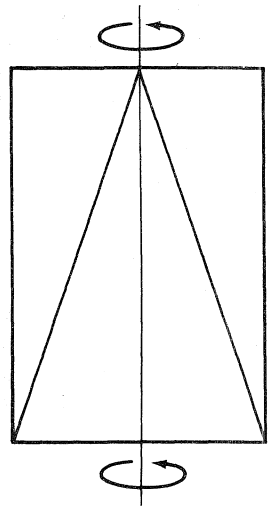

Fig. 3

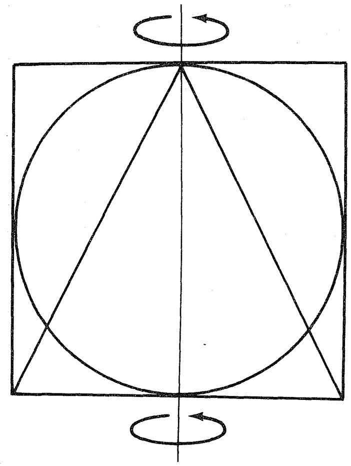

Fig. 4

Some years after my teacher, George Pólya, had shown me that result, I read about the researches in aesthetics conducted by Gustav Theodor Fechner (in Germany), whose experimental results in 1876—later confirmed in varying degrees by others—indicated that 75.6 percent (!) of his popular observers of rectangles found that rectangle to be most pleasant whose proportions are about 8 : 5, as in Figure 1. The "about 8 : 5" refers to what Renaissance writers referred to as the "divine proportion," the "golden mean" or "golden section" of Greek geometers, used by da Vinci and others, by Salvador Dali in our time, and still making its appearance in some contemporary design. (The façade of the ancient Parthenon, if one includes the face of the roof, fits into such a rectangle. Dali's "Last Supper" painting has exactly these proportions.)

Now 8 ÷ 5 = 1.60, while the "golden mean" is actually

$$(1 + \sqrt{5}) \div 2 = 1.6180339\ldots,$$

a number which solves the equation $x^2 - x - 1 = 0$ and is well known to Fibonacci people. This equation arises when geometers divide a line segment in "extreme and mean ratio"; i.e., so that its length is divided into parts of length x and 1 such that $(x + 1)/x = x/1$.

Figure 5 shows how easily we can construct a rectangle which possesses that "most pleasant shape." We simply start with a square, *ABCD*, and locate *M*, the midpoint of its base. With the length *MC* from *M* to an opposite corner *C* as a radius, we locate the point *P* on the extension of *AB* shown. The sides of the resulting big rectangle *APQD* have lengths which are as $1 + \sqrt{5}$ is to 2.

Well, after having read about conclusions such as Fechner's, it once (quite long ago now) occurred to me to draw what I have here shown as Figure 1, involving that most pleasant rectangular shape, and to calculate the volumes of revolution generated by spinning *this* figure about its vertical bisecting axis. This can very *quickly* be done with the powerful elementary tools of calculus which have been bequeathed to us. I was then privileged to encounter the following beautiful result about these *volumes*:

(β) Cone : Ellipsoid : Cylinder = 1 : 2 : 3.

(If the cone holds one liter, then the ellipsoid holds two liters, and the cylindrical can will hold precisely 3 liters.)

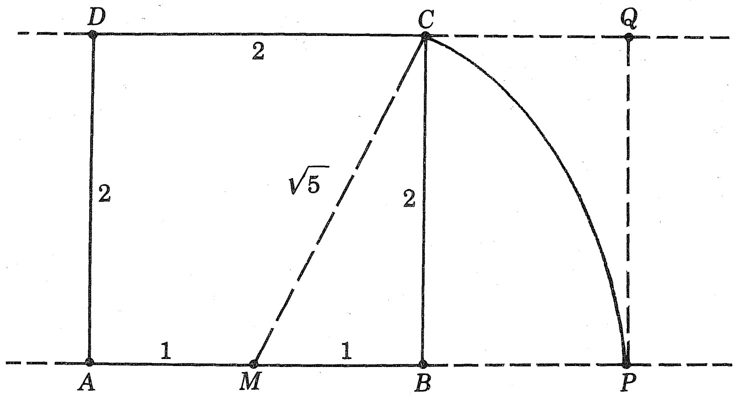

Fig. 5

I don't think Archimedes knew this theorem. I am sure he would have treasured it. And if he had not been killed by a Roman soldier while he was absorbed in studying some circles in his home (in Syracuse, in southern Sicily, during the second Punic war), he might well have observed it and recorded it at a later time.

[Here are the *calculations* which yield the result (β). We refer to Figure 1, where we let the radius of revolution be r, and the height of the cylinder be h. Then

$$\text{CONE volume} = \frac{\text{(BASE area)} \times \text{(height)}}{3} = \pi r^2 h/3,$$

and, of course, the volume of the CYLINDER, $\pi r^2 h$, is exactly three times this number.

The volume of an ellipsoid with minor axes of length a, b, and c, is

$$(4/3)\pi abc.$$

Our ELLIPSOID from Figure 1 thus has volume equal to

$$(4/3)\pi(r)(r)(h/2) = (2/3)\pi r^2 h.$$

Putting all of this together, we have this relation among the three volumes of revolution:

$$\text{CONE} : \text{ELLIPSOID} : \text{CYLINDER} = 1/3 : 2/3 : 3/3 = 1 : 2 : 3.$$

These calculations show that, actually, this beautiful result (β) holds for *any* encompassing rectangular shape—not just one with divine proportions. Furthermore, it should be recorded here that this result, once we have *guessed* it, is directly derivable from (α) by an application of the powerful (modern) theory of "affine transformations."]

It is this result (β), then, which is built into the sculpture now in our Quad, installed as a tribute back over the ages to Archimedes. (The rectangle in our San Jose State sculpture is about 1.5 feet across, by the way, and the number associated with its proportions is about 1.61, which is, we think, close enough for anybody riding by on a horse!)

Finally, one of the speakers at the January 19, 1981 dedication ceremonies was Professor Gerald Alexanderson (a friend of Dr. Hoggatt's and mine, and Chairman of the Mathematics Department at the nearby University of Santa Clara), who spoke on behalf of Dr. Dorothy Bernstein, President of The Mathematical Association of America. Here are a few excerpts from his remarks about "this powerful piece of sculpture honoring Archimedes."

"As one who has spent much time and money trying to locate and visit mathematical shrines, often in the form of statuary and monuments to mathematicians, I am particularly happy to be here today. Let us review what some of those monuments are. There's the statue of Simon Stevin in Bruges. (For those whose history of science is a little rusty: he gave us decimals.) Then there are those great cenotaphs for the Bernoullis in the Peterskirche in Basel, with wonderful ladies in marble doing geometry with golden compasses. Of course, the best part of a visit to the Peterskirche is that one walks up the Eulerstrasse to get there. There's the Gauss-Weber monument in Göttingen. Actually, I think they're shown doing physics, but never mind. Gauss was certainly a mathematician. (They are discussing their invention of the telegraph.) And a favorite of many is Roubiliac's statue of Newton outside the chapel at Trinity College, Cambridge. But the best of all is the romantic, heroic statue of Abel in the Royal Park in Oslo. He is shown standing erect, head thrown back, with hair caught in the wind, and he's standing on two vanquished figures, obviously beaten in battle. One is the elliptic function and the other is the fifth-degree polynomial equation. Actually, I cannot tell which is which, because they're not terribly good likenesses.

"Now right here in San Jose we have a monument to Archimedes. I am grateful. When the urge comes on to visit a mathematical monument, it will be much more convenient (and cheaper) to make a pilgrimage to San Jose, than to Syracuse."

A BRIEF ANNOTATED BIBLIOGRAPHY

(A short note having the same title as this present one appears on page 339 of the May 1981 issue of the *American Mathematical Monthly*.)

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(Archimedes' "Method of Exhaustion" briefly treated.)

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(Professor Bell, discussing Zeno, Eudoxus, and Archimedes—"modern minds in ancient bodies"—suggests that Archimedes' "Give me a place to stand and I will move the earth" would make an excellent motto for a modern scientific institute.)

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(Key, applicable information about affine transformations pulled together in a convenient package appears in this paper.)

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(The method of exhaustion and the derivation of the volume of a sphere as Archimedes did it.)

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(Discussion of the surface area and volume ratios mentioned in connection with Figure 2 above. Cicero writes about the Archimedes gravestone around 75 B.C.)

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(Chapter about Gustav Fechner, who "founded experimental esthetics." His first paper in this new field was on the golden section, and appeared in 1865.)

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(The earliest appearance of the term "divine proportion"?)

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(A table somewhat related to our Figure 4 above, with references to Cavalieri's principle and to the rules of Pappus for calculating volumes.)

________. *Mathematics and Plausible Reasoning*, Vol. I, pp. 155-158. Princeton, 1954.
(Derivation of the volume of a sphere by Archimedes. Professor Pólya tells us about the work of Democritus. It was Pólya who first told me about the simple yet very powerful *Axiom of Archimedes*, which figures so prominently in some of the work of Archimedes. It says simply this: If a and b are any given positive lengths, then it is always possible to take enough copies of a, say na, so that the length na exceeds the length b.)

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(Lots of references to the golden section. References to classical Renaissance paintings and various aspects of contemporary design. ". . . the majority of people consider the most aesthetically pleasing shape that rectangle whose sides are in the approximate ratio 8 : 5.")

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THE UBIQUITOUS RATIONAL SEQUENCE

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DEDICATION

Vern Hoggatt has been the inspiration of many papers that have appeared in this journal. He shared his enthusiasm and curiosity about mathematics with a notable generosity. His students, friends, and pen pals were enriched by the problems he posed and often helped to solve. My own interest in sequences was greatly influenced by the correspondence we started when I was a graduate student at the University of Alberta. Some of my first papers written at that