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SOME DIVISIBILITY PROPERTIES OF PASCAL'S TRIANGLE

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This paper is dedicated to the memory of Professor V. E. Hoggatt, Jr., whose happy enthusiasm for mathematics has been an inspiration to all who knew him and whose friendship has enormously enriched the lives of so many, including, in particular, the present author.

1. INTRODUCTION

Let p denote a prime and let m, n, h, k, and α denote integers with

$$0 \le k \le n,\ 1 \le h \le n,\ m \ge 1, \text{ and } \alpha \ge 1.$$

Let $\Delta_{n,k}$ denote the triangle of entries

$$\begin{array}{ccc} & \binom{nm}{km} & \\ \cdot^{\cdot^{\cdot}} & & \ddots \\ \binom{nm+m-1}{km} & \cdots & \binom{nm+m-1}{km+m-1} \end{array}$$

from Pascal's triangle. And let $\nabla_{n,h}$ denote the triangle of entries from Pascal's triangle indicated by

$$\begin{array}{ccc} \binom{nm}{hm-m+1} & \cdots & \binom{nm}{hm-1} \\ \ddots & & \cdot^{\cdot^{\cdot}} \\ & \binom{nm+m-2}{hm-1} & \end{array}$$

For $m = p^{\alpha}$, we showed in [2] that all elements of Pascal's triangle not contained in some $\Delta_{n,k}$ (i.e., those contained in some $\nabla_{n,h}$) are congruent to 0 modulo p, that, modulo p, there are precisely p distinct triangles $\Delta_{n,k}$, and that these triangles can be put in one-to-one correspondence with the residues 0, 1, 2, ..., $p - 1$ in such a way that the triangle of triangles

$$\begin{array}{ccccc} & & \Delta_{0,0} & & \\ & \Delta_{1,0} & & \Delta_{1,1} & \\ \Delta_{2,0} & & \Delta_{2,1} & & \Delta_{2,2} \\ & & \cdots\cdots\cdots & & \end{array}$$

is "isomorphic" to the original Pascal triangle in the sense that

$$\Delta_{n,k} + \Delta_{n,k+1} = \Delta_{n+1,k+1}$$

where the addition is elementwise modulo p. We also showed that if D is the greatest common divisor of the three corner elements of $\nabla_{n,1}$ and d is the greatest common divisor of all the elements of $\nabla_{n,1}$, then $d = p$ and $D = p^\alpha$ if $m = p^\alpha$ and $d = 1$ and $D = m$ for all other integers $m \geq 2$. In the present paper we obtain, for $m = p^2$, a result similar to the first result for $\Delta_{n,k}$ and, for $m = p^\alpha$, we extend the second result to $\nabla_{n,h}$ for $1 \leq h \leq n$. Finally, we obtain a number of interesting properties of the p-index triangle of Pascal's triangle, which is simply the triangle of numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ that indicate the exponent e to which a given prime p divides $\binom{n}{k}$; i.e., such that $p^e \,\Big|\, \binom{n}{k}$ and $p^{e+1} \nmid \binom{n}{k}$.

2. THE ITERATED TRIANGLE MODULO p^2

To extend the first result mentioned above to p^2, we set $\alpha = 2\beta$ where $\beta \geq 1$ is an integer. Thus, the $\Delta_{n,k}$ are equilateral triangles with $p^{2\beta}$ elements per side. Furthermore, we say that two such triangles are equivalent provided that their top p rows are identical, and it is clear that this is an equivalence relation in the technical sense. Let $\delta_{n,k}$ denote the class of all triangles equivalent to $\Delta_{n,k}$. Then, again, we claim that there exist precisely p^2 equivalence classes of triangles, and that there exists a one-to-one correspondence between these classes and the residues $0, 1, \ldots, p^2 - 1$ such that the triangular array

$$\begin{array}{ccccc} & & \delta_{0,0} & & \\ & \delta_{1,0} & & \delta_{1,1} & \\ \delta_{2,0} & & \delta_{2,1} & & \delta_{2,2} \\ & & \cdots\cdots\cdots & & \end{array}$$

is "isomorphic" to the original Pascal triangle in the sense that

$$\delta_{n,k} + \delta_{n,k+1} = \delta_{n+1,k+1}$$

where the addition is defined by the elementwise addition modulo p^2 of the top p rows of any two representatives of $\delta_{n,k}$ and $\delta_{n,k+1}$. All of this follows from Theorem 5 below, but first we need several lemmas. The first is well known (see, for example, [1, problem 16, p. 57]).

Lemma 1: Let p be a prime and let n and k be integers with $0 \leq k \leq n$. Then

$$p^e \,\Big\|\, \binom{n}{k}$$

if and only if e is the number of carries made in adding k to $n - k$ in base p. Equivalently, e is the number of carries made in subtracting k from n in base p.

Lemma 2: Let p be a prime and let k, h, and α be integers with $k \geq 1$, $\alpha \geq 1$, and $0 \leq h \leq kp^\alpha$. If $p \nmid h$, then

$$\binom{kp^\alpha}{h} \equiv 0 \pmod{p^\alpha}.$$

Proof: Since $p \nmid h$, the units digit in the base p representation of h is not zero. Thus, it is clear that the subtraction of h from kp^α in base p requires α carries and the result follows from Lemma 1.

We note in passing that the converse of Lemma 2 is false since, for example,

$$\binom{16}{2} \equiv 0 \pmod 4$$

and yet $2|2$.

Lemma 3: Let p be a prime and let k, h, α, and β be integers with $\beta \geq 1$, $\alpha = 2\beta$, and $0 \leq h \leq k$. Then

$$\binom{kp^\alpha}{hp^\alpha} \equiv \binom{k}{h} \pmod{p^2}.$$

Proof: In [4], J. H. Smith proves that

$$\binom{kp}{hp} \equiv \binom{k}{h} \pmod{p^2}.$$

Thus, the result claimed follows immediately by induction.

Lemma 4: Let p be a prime and let k, h, r, s, α, and β be integers with

$$0 \leq h \leq k,\ 0 \leq s \leq r < p,\ \beta \geq 1, \text{ and } \alpha = 2\beta.$$

Then

$$\binom{kp^\beta + r}{hp^\beta + s} \equiv \binom{k}{h}\binom{r}{s} \pmod{p^2}.$$

Proof: This is an immediate consequence of Lemma 3 and the fact that, by Lemma 2, the $p - 1$ coefficients on either side of $\binom{kp^\beta}{hp^\beta}$ must all be congruent to 0 modulo p^2.

Theorem 5: Let p be a prime and let α, β, k, and n be integers with $\beta \geq 1$, $\alpha = 2\beta$, and $0 \leq k \leq n$. Then the first p rows of $\Delta_{n,k}$ modulo p^2 are

$$\begin{array}{ccccc}
 & & \binom{n}{k}\binom{0}{0} & & \\
 & \binom{n}{k}\binom{1}{0} & & \binom{n}{k}\binom{1}{1} & \\
 \cdot^{\cdot^{\cdot}} & & & & \ddots \\
\binom{n}{k}\binom{p-1}{0} & & \cdots & & \binom{n}{k}\binom{p-1}{p-1}
\end{array}$$

Also,

$$\delta_{n,k} + \delta_{n,k+1} = \delta_{n+1,k+1}$$

where $\delta_{n,k}$ and this addition are defined above.

Proof: The elements in the first p rows of $\Delta_{n,k}$ are the binomial coefficients

$$\binom{np^\beta + r}{kp^\beta + s},\ 0 \leq s \leq r < p,$$

and, by Lemma 4,

$$\binom{np^\beta + r}{kp^\beta + s} \equiv \binom{n}{k}\binom{r}{s} \pmod{p^2}.$$

This gives the first assertion of the theorem and implies the second since

$$\binom{np^\beta + r}{kp^\beta + s} + \binom{np^\beta + r}{(k+1)p^\beta + s} \equiv \binom{n}{k}\binom{r}{s} + \binom{n}{k+1}\binom{r}{s}$$

$$\equiv \binom{n+1}{k+1}\binom{r}{s} \equiv \binom{(n+1)p^\beta + r}{(k+1)p^\beta + s} \pmod{p^2}.$$

Of course, the fact that every entry in Pascal's triangle not contained in $\Delta_{n,k}$ for $m = p^\alpha = p^{2\beta}$ is congruent to zero modulo p follows immediately from Theorem 1 of [2] with $\alpha = 2\beta$. One might have guessed that all these elements were in fact congruent to zero modulo p^2, but this is easily seen not to be the case. In particular, $\binom{4}{2}$ is not contained in any $\Delta_{n,k}$ for $p = 2$ and $\alpha = 2$, and

$$0 \not\equiv \binom{4}{2} \equiv 2 \pmod{4}.$$

3. SOME GREATEST COMMON DIVISOR PROPERTIES

Recall that for integers m, n, and h with $1 \le m$ and $1 \le h \le n$, $\nabla_{n,h}$ denotes the triangle of binomial coefficients

$$\begin{matrix} \binom{nm}{hm - m + 1} & & \cdots & & \binom{nm}{hm - 1} \\ & \cdot & & \cdot & \\ & & \binom{nm + m - 2}{hm - 1}, & & \end{matrix}$$

that d denotes the greatest common divisor of all the coefficients of $\nabla_{n,h}$, and that D denotes the greatest common divisor of the corner coefficients. In [2], we completely determined d and D for $\nabla_{n,1}$ and we extend those results in this section. The increased generality, however, makes for somewhat weaker results as seen in the following theorems.

Theorem 6: Let d, D, and $\nabla_{n,h}$ be as above where $m = p^\alpha$ with p a prime and α a positive integer. If $p^e \| n$, then $p^{e+1} \| d$ and $p^{e+\alpha} \| D$.

Proof: We first prove that $p^{e+\alpha} \| D$. The upper left coefficient in $\nabla_{n,h}$ is

$$L = \binom{np^\alpha}{(h - 1)p^\alpha + 1}.$$

Since $p^e \| n$, $np^\alpha = Np^{e+\alpha}$ where $p \nmid N$. Also, the units digit in the base p representation of $(h - 1)p + 1$ is 1 and this clearly implies that $e + \alpha$ carries are required in subtracting $(h - 1)p^\alpha + 1$ for np^α in base p. Thus, $p^{e+\alpha} \| L$. The upper right coefficient in $\nabla_{n,h}$ is

$$R = \binom{np^\alpha}{hp^\alpha - 1}.$$

Since $p \nmid hp^\alpha - 1$, the units digit in the base p representation of $hp^\alpha - 1$ is not 0 and again $e + \alpha$ carries are required in subtracting $hp^\alpha - 1$ from np^α in base p. Thus, $p^{e+\alpha} \| R$. Finally, the bottom coefficient of $\nabla_{n,h}$ is

$$B = \binom{(n + 1)p^\alpha - 2}{hp^\alpha - 1}.$$

Here,

$$\begin{aligned} (n + 1)p^\alpha - 2 &= np^\alpha + p^\alpha - 2 \\ &= Np^{e+\alpha} + (p - 1)p^{\alpha - 1} + \cdots + (p - 1)p + (p - 2), \\ hp^\alpha - 1 &= (h - 1)p^\alpha + p^\alpha - 1 \\ &= (h - 1)p^\alpha + (p - 1)p^{\alpha - 1} + \cdots + (p - 1)p + (p - 1), \end{aligned}$$

and, since $p - 2 < p - 1$, it follows that the subtraction of $hp^\alpha - 1$ from $(n + 1)p^\alpha - 2$ necessitates $e + \alpha$ carries. Thus, $p^{e+\alpha} \| B$ and it follows that $p^{e+\alpha} \| D$ as claimed.

To show that $p^{e+1} \| d$, it suffices to show that p^{e+1} divides each element in the top row of $\nabla_{n,h}$ and that p^{e+1} exactly divides one of these elements. The elements in the top row of $\nabla_{n,h}$ are

$$\binom{np^\alpha}{hp^\alpha - p^\alpha + s}, \quad 1 \le s \le p^\alpha - 1.$$

Again $np^\alpha = Np^{e+\alpha}$ where $p \nmid N$. Since $1 \le s \le p^{\alpha+1}$, the base p representation of s must contain at least one nonzero digit in some position prior to the α-th. Thus, the subtraction of $hp^\alpha - p^\alpha + s$ from np^α requires carries from the $(e + \alpha)$ column of the base p representation of np^α and these must be at least $e + 1$ in number. Thus, p^{e+1} divides every element in the top row of $\nabla_{n,h}$. Now consider the element

$$M = \binom{np^\alpha}{hp^\alpha - p^\alpha + p^{\alpha-1}}.$$

Again, since $np^\alpha = Np^{e+\alpha}$ as above, the carrying in the subtraction of $(h - 1)p^\alpha + p^{\alpha-1}$ from np^α is precisely from the $e + \alpha$ column to the $\alpha - 1$ column for a total of exactly $e + 1$ carries. Therefore, $p^{e+1} \| M$ and $p^{e+1} \| d$ as claimed.

Note that for $p = 3$ and $\alpha = 2$, $\nabla_{4,2}$ is such that

$$d = 16{,}182 = 2 \cdot 3^2 \cdot 29 \cdot 31 \quad \text{and} \quad D = 48{,}546 = 3d$$

and this suggests that Theorem 6 might be considerably strengthened. However, $d = 3$ and $D = 3^2$ in $\nabla_{4,1}$. Also, if $n = h = 1$, p is a prime, and α is a positive integer, then $d = p$ and $D = p^\alpha$ by Theorem 2 of [2]. Thus, in a sense, Theorem 6 is best possible for prime powers.

In case m is composite but not a prime power, our best result is as follows.

Theorem 7: Let d, D, and $\nabla_{n,h}$ be as above with m composite and not a prime power. Then $m | D$.

Proof: Let $p^\alpha \| m$ so that $m = Mp^\alpha$ and $p \nmid M$. Then the argument of Theorem 6 can be repeated exactly to thow that $p^\alpha | D$. Thus, if

$$m = \prod_{i=1}^{r} p_i^{\alpha_i}$$

is the canonical representation of m, it follows that $p_i^{\alpha_i} | D$ for each i and hence that $m | D$ as claimed.

Several examples suffice to show that Theorem 7 is also, in a sense, best possible. Consider the triangles $\nabla_{n,h}$ with $m = 6$. That d is not necessarily equal to 1 even when $6 \nmid n$ is shown by $\nabla_{5,2}$ where $d = 870$. Also, the fact that $m | n$ does not necessarily increase the power of m that divides d and D is shown by $\nabla_{6,1}$ where $d = 3$ so that $6 \nmid d$ and by $\nabla_{6,2}$ where $6 | D$ but $6^2 \nmid D$.

4. THE p-INDEX TRIANGLE

For a given prime p, let $\left[\begin{matrix} n \\ k \end{matrix}\right]$ denote the exponent of the highest power of p that divides $\binom{n}{k}$. The triangle of entries $\left[\begin{matrix} n \\ k \end{matrix}\right]$, $0 \le k \le n$, is called the p-index triangle of Pascal's triangle and seems to have been studied first by K. R. McLean [3]. Quite apart from their attractiveness as kind of mathematical art, the p-index triangles exhibit interesting patterns that reveal additional structure in Pascal's triangle. Some of the more interesting of these properties are detailed in the following theorems.

Theorem 8: Let p be a prime, then

$$\begin{bmatrix} mp^\alpha - 1 \\ k \end{bmatrix} = 0 \text{ for } 1 \le m < p \text{ and } 0 \le k \le mp^\alpha - 1.$$

Proof: Since $0 \le k \le mp^\alpha - 1$ and $1 \le m < p$,

$$k = \sum_{i=0}^{\alpha} k_i p^i$$

with $0 \le k_i < p$ for all i and $k_\alpha < m$. But

$$mp^\alpha - 1 = (m - 1)p^\alpha + \sum_{i=0}^{\alpha-1} (p - 1)p^i.$$

Thus, there are no carries in subtracting k from $mp^\alpha - 1$ in base p and

$$\begin{bmatrix} mp^\alpha - 1 \\ k \end{bmatrix} = 0$$

as claimed.

Theorem 9: Let p be a prime, then $\begin{bmatrix} p^\alpha \\ k \end{bmatrix} \ge 1$ for $1 \le k < p^\alpha$. Of course,

$$\begin{bmatrix} p^\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} p^\alpha \\ p^\alpha \end{bmatrix} = 0.$$

Proof: Since $1 \le k < p^\alpha$,

$$k = \sum_{i=0}^{\alpha-1} k_i p^i$$

with $0 \le k_i < p$ for all i and $k_i \ne 0$ for at least one i. Therefore, there is at least one carry in subtracting k from p^α and the result follows.

Theorem 10: Let p be a prime and let m and n be positive integers with $1 \le m < p$ and $1 \le n < p$. Then

$$\begin{bmatrix} mp^\alpha + np^{\alpha-1} - 1 \\ k \end{bmatrix} = \begin{cases} 0 \text{ for } rp^\alpha \le k \le rp^\alpha + np^{\alpha-1} - 1, \ 0 \le r \le m \\ 1 \text{ for } rp^\alpha + np^{\alpha-1} \le k \le (r + 1)p^\alpha, \ 0 \le r < m. \end{cases}$$

Proof: Note that

$$mp^\alpha + np^{\alpha-1} - 1 = mp^\alpha + (n - 1)p^{\alpha-1} + \sum_{i=0}^{\alpha-2} (p - 1)p^i.$$

Thus, if $0 \le k \le mp^\alpha + np^{\alpha-1} - 1$, the only time a carry will be required in subtracting k from $mp^\alpha + np^{\alpha-1} - 1$ in base p is when k has a digit $k_{\alpha-1} \ge n$ in the $\alpha - 1$ position. But this occurs precisely when

$$np^{\alpha-1} \le k < p^\alpha, \text{ or } p^\alpha + np^{\alpha-1} \le k < 2p^\alpha, \text{ or } \ldots,$$

or $(m - 1)p^\alpha + np^{\alpha-1} \le k < mp^\alpha$ as claimed.

As in Pascal's triangle modulo p, the p-index triangle naturally decomposes into an array of interesting subtriangles. Thus, we have the following results.

Theorem 11: Let p be a prime and let $\alpha \geq 1$ be an integer. For integers n and k with $0 \leq k \leq n$, let $T_{n,k}$ denote the subtriangle of entries from the p-index triangle indicated by

$$\begin{array}{ccc} & \begin{bmatrix} np^\alpha \\ kp^\alpha \end{bmatrix} & \\ & \cdot \quad \cdot & \\ \begin{bmatrix} (n+1)p^\alpha - 1 \\ kp^\alpha \end{bmatrix} & \cdots & \begin{bmatrix} (n+1)p^\alpha - 1 \\ (k+1)p^\alpha - 1 \end{bmatrix} \end{array}$$

Then, $T_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix} + T_{0,0}$ where this is understood to mean that $\begin{bmatrix} n \\ k \end{bmatrix}$ is added to each element of $T_{0,0}$.

Proof: Note that $T_{0,0}$ is the triangle of entries

$$\begin{bmatrix} r \\ s \end{bmatrix}, \ 0 \leq s \leq r < p^\alpha.$$

Similarly, $T_{n,k}$ is the triangle of entries

$$\begin{bmatrix} np^\alpha + r \\ kp^\alpha + s \end{bmatrix}, \ 0 \leq s \leq r < p^\alpha.$$

Since $s \leq r$, the number of carries required in subtracting $kp^\alpha + s$ from $np^\alpha + r$ is just the number required in subtracting k from n plus those required in subtracting s from r. That is

$$\begin{bmatrix} np^\alpha + r \\ kp^\alpha + s \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} r \\ s \end{bmatrix}$$

Thus, $T_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix} + T_{0,0}$ as claimed.

Corollary 12: Consider the infinite array of triangles $T_{n,k}$, $0 \leq k \leq n$, as in Theorem 11. The array consisting of the top vertex element of each of these triangles is just the original p-index triangle. Thus, the p-index triangle contains a p-index triangle which contains a p-index triangle, and so on without end.

Proof: The triangle of top elements of $T_{n,k}$ is the triangle $\begin{bmatrix} np^\alpha \\ kp^\alpha \end{bmatrix}$, $0 \leq k \leq n$. But, by Theorem 11,

$$\begin{bmatrix} np^\alpha \\ kp^\alpha \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}$$

and the result follows.

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