ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A. P. HILLMAN
University of Santa Clara, Santa Clara, California

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. We welcome any problems believed to be new in the area of recurrent sequences as well as new approaches to existing problems. The proposer should submit his problem with solution in legible form, preferably typed in double spacing, with name(s) and address of the proposer clearly indicated.

Solutions to problems listed below should be submitted within two months of publication.

B-30 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

Find the millionth term of the sequence a_n given that

$a_1 = 1$, $a_2 = 1$, and $a_{n+2} = a_{n+1} - a_n$ for $n \geq 1$.

B-31 *Proposed by Douglas Lind, Falls Church, Virginia*

If n is even, show that the sum of $2n$ consecutive Fibonacci numbers is divisible by F_n.

B-32 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas.*

Show that $nL_n \equiv F_n \pmod 5$.

B-33 *Proposed by John A. Fuchs, University of Santa Clara, Santa Clara, California*

Let $u_n, v_n, \cdots, w_n$ be sequences each satisfying the second order recurrence formula

$$y_{n+2} = gy_{n+1} + hy_n \quad (n \geq 1) ,$$

where g and h are constants. Let $a, b, \cdots, c$ be constants. Show that

$$au_n + bv_n + \cdots + cw_n = 0$$

is true for all positive integral values of n if it is true for $n = 1$ and $n = 2$.

B-34 *Proposed by G. L. Alexanderson, University of Santa Clara, Santa Clara, California*

Let u_n and v_n be any two sequences satisfying the second order recurrence formula

$$y_{n+2} = gy_{n+1} + hy_n$$

where g and h are constants. Show that the sequence of products $w_n = u_n v_n$ satisfies a third-order recurrence formula

$$y_{n+3} = ay_{n+2} + by_{n+1} + cy_n$$

and find a, b, and c as functions of g and h.

B-35 *Proposed by J. L. Brown, Jr., Pennsylvania State University, University Park, Pa.*

Prove that

$$\sum_{k=1}^{r-1} (-1)^k \binom{r}{k} F_k = 0$$

for r an odd positive integer and generalize.

B-36 *Proposed by Roseanna Torretto, University of Santa Clara, Santa Clara, California*

The sequence $1, 2, 5, 12, 29, 70, \cdots$ is defined by $c_1 = 1$, $c_2 = 2$, and $c_{n+2} = 2c_{n+1} + c_n$ for all $n \geq 1$. Prove that c_{5m} is an integral multiple of 29 for all positive integers m.

B-37. *Proposed by Brother U. Alfred, St. Mary's College, California*

Given a line with a point of origin O and four positive positions A, B, C, and D with respect to O. If the line segments OA, OB, OC, and OD correspond respectively to four consecutive Fibonacci numbers $F_n, F_{n+1}, F_{n+2}, F_{n+3}$, determine for which set(s) of Fibonacci numbers the points A, B, C, and D are in simple harmonic ratio, i.e.,

$$\frac{AB}{BC}\,\frac{AD}{DC} = -1 .$$

SOLUTIONS

DIFFERENCES MADE INTO PRODUCTS

B-17 *Proposed by Charles R. Wall, Ft. Worth, Texas*

If m is an integer, prove that

$$F_{n+4m+2} - F_n = L_{2m+1} F_{n+2m+1}$$

where F_p and L_p are the p^{th} Fibonacci and Lucas numbers, respectively.

Solution by I. D. Ruggles, San Jose State College, San Jose, California

In "Some Fibonacci Results Using Fibonacci-Type Sequences," Fibonacci Quarterly, Vol. 1, No. 2, p. 77, it is shown that

$$F_{q+p} - F_{q-p} = L_p F_q, \quad \text{for } p \text{ odd.}$$

If $q = n + 2m + 1$ and $p = 2m + 1$, then this becomes the desired formula.

Also solved by Douglas Lind, Falls Church, Virginia, and the proposer.

A TRIGONOMETRIC SUM

B-18 *Proposed by J. L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania.*

Show that

$$F_n = 2^{n-1} \sum_{k=0}^{n-1} (-1)^k \cos^{n-k-1} \frac{\pi}{5} \sin^k \frac{\pi}{10} \quad \text{for} \quad n \geq 0.$$

(It should be "for $n \geq 1$" instead of "for $n \geq 0$.")

Solution by the proposer

It is well known (e.g., I. J. Schwatt, "An Introduction to the Operations with Series," Chelsea Pub. Co., p. 177) that

$$\cos \frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}$$

$$\cos \frac{\pi}{10} = \frac{\sqrt{5} - 1}{4}$$

Therefore,

$$a = \frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5},$$

$$b = \frac{1-\sqrt{5}}{2} = -2\sin\frac{\pi}{10}$$

and

$$F_n = \frac{a^n - b^n}{a-b} = 2^{n-1}\left[\frac{\cos^n\frac{\pi}{5} - (-1)^n\sin^n\frac{\pi}{10}}{\cos\frac{\pi}{5} + \sin\frac{\pi}{10}}\right]$$

$$= 2^{n-1}\frac{\cos^n\frac{\pi}{5} - \sin^n\left(-\frac{\pi}{10}\right)}{\cos\frac{\pi}{5} - \sin\left(-\frac{\pi}{10}\right)} = 2^{n-1}\sum_{k=0}^{n-1}\cos^{n-k-1}\frac{\pi}{5}\sin^k\left(-\frac{\pi}{10}\right)$$

$$= 2^{n-1}\sum_{k=0}^{n-1}(-1)^k\cos^{n-k-1}\frac{\pi}{5}\sin^k\frac{\pi}{10}$$

as stated. We have made use of the algebraic identity

$$\frac{x^n - y^n}{x-y} = \sum_{k=0}^{n-1} x^{n-k-1}y^k .$$

Also solved by Charles R. Wall, Texas Christian University, who pointed out that the identity does not hold for $n=0$.

A TELESCOPING SUM

B-19 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Show that

$$\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2}^2 F_{n+3}} + \sum_{n=1}^{\infty}\frac{1}{F_n F_{n+1}^2 F_{n+3}} = \frac{1}{2}$$

Solution by John H. Avila, University of Maryland, College, Park Maryland

Our solution is similar to that by Francis D. Parker for B-9. Let $a = a(n) = F_n$, $b = F_{n+1}$, $c = F_{n+2}$, and $d = F_{n+3}$. Then $a + b = c$, $b + c = d$, and the left side of the desired formula is

$$\sum_{n=1}^{\infty}\left(\frac{1}{ac^2d}+\frac{1}{ab^2d}\right) = \sum_{n=1}^{\infty}\left(\frac{b}{abc^2d}+\frac{c}{ab^2cd}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{c-a}{abc^2d}+\frac{d-b}{ab^2cd}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{abcd}-\frac{1}{bc^2d}+\frac{1}{ab^2c}-\frac{1}{abcd}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{ab^2c}-\frac{1}{bc^2d}\right) .$$

The last sum is the telescoping series

$$\left(\frac{1}{F_1F_2^2F_3}-\frac{1}{F_2F_3^2F_4}\right)+\left(\frac{1}{F_2F_3^2F_4}-\frac{1}{F_3F_4^2F_5}\right)+\cdots$$

whose sum is

$$\frac{1}{F_1F_2^2F_3} = \frac{1}{1\cdot 1^2\cdot 2} = \frac{1}{2} .$$

Also solved by the proposer.

SUMMING GENERALIZED FIBONACCI NUMBERS

B-20 *Proposed by Louis G. Brökling, Redwood City, California*

Generalize the well-known identities,

(i) $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$

(ii) $L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 3$.

Solution by Charles R. Wall, Texas Christian University, Ft. Worth, Texas

If $H_0 = q$, $H_1 = p$, and $H_{n+2} = H_{n+1} + H_n$, then $H_n = pF_n + qF_{n-1}$ so that

$$\sum_{i=1}^{n} H_i = p\sum_{i=1}^{n} F_i + q\sum_{i=0}^{n-1} F_i = p(F_{n+2} - 1) + q(F_{n+1} - 1)$$

$$= pF_{n+2} + qF_{n+1} - (p+q) = H_{n+2} - (p+q) = H_{n+2} - H_2 .$$

This identity is also obtained from Horadam's "A Generalized Fibonacci Sequence," American Mathematical Monthly, Vol. 68 (1961), p. 456.

Also solved by Fern Grayson, Lockheed Missiles and Space Company, Sunnyvale California and the proposer.

EVENS AND ODDS

B-21 *Proposed By L. Carlitz, Duke University, Durham, N. C.*

If

$$u_n = \frac{1}{2}\left[(x+1)^{2^n} + (x-1)^{2^n}\right]$$

show that

$$u_{n+1} = u_n^2 + 2^{2n} u_0^2 u_1^2 \cdots u_{n-1}^2 .$$

Solution by Robert Means, University of Michigan

Let Let $v_n = (x+1)^{2^n} - u_n$. Then $u_0 = x$, $v_0 = 1$ and for $n \geq 1$ u_n and v_n are the terms of even and of odd degree respectively in $(x+1)^{2^n}$. Now $u_{n+1} + v_{n+1} = (u_n + v_n)^2 = u_n^2 + 2u_n v_n + v_n^2$ and equating sums of terms of even and of odd degree respectively we have for $n \geq 0$,

(a) $$u_{n+1} = u_n^2 + v_n^2$$

(b) $$v_{n+1} = 2u_n v_n .$$

Repeated use of (b) leads to $v_n = 2u_{n-1}v_{n-1} = 2^2u_{n-1}u_{n-2}v_{n-2} = \cdots = 2^n u_{n-1}u_{n-2} \cdots u_0 v_0$. . Since $v_0 = 1$, the desired result is obtained by substituting the last expression for v_n in (a).

Also solved by Charles R. Wall, Texas Christian University and the proposer

LUCAS ANALOGUES

B-22 *Proposed by Brother U. Alfred, St. Mary's College, California*

Prove the Fibonacci identity

$$F_{2k}F_{2k'} = F^2_{k+k'} - F^2_{k-k'}$$

and find the analogous Lucas identity.

(Editor's Note: The Fibonacci identity here is proved by I. D. Ruggles in "Some Fibonacci Results Using Fibonacci-Type Sequences," this Quarterly, Vol. 1, Issue 2, p. 77.) Proofs were submitted by Douglas Lind, Falls Church, Virginia; V. E. Hoggatt, Jr., San Jose State College; and Charles R. Wall, Texas Christian University, Ft. Worth, Texas. Lind and Hoggatt gave

$$L_{2k}L_{2j} = L^2_{k+j} + L^2_{k-j} - 4(-1)^{k-j}$$

as the analogous Lucas identity and Wall gave it as

$$L_{2k}L_{2j} = L^2_{k+j} + 5F^2_{k-j} = 5F^2_{k+j} + L^2_{k-j} \quad .$$

Proofs of these are left to the readers.

TELESCOPING PRODUCTS AND SUMS

B-23 *Proposed by S. L. Basin, Sylvania Electronic Systems, Mt. View, Calif.*

Prove the identities

(i) $$F_{n+1} = \prod_{i=1}^{n}\left(1 + \frac{F_{i-1}}{F_i}\right)$$

(ii) $$\frac{F_{n+1}}{F_n} = 1 + \sum_{i=2}^{n} \frac{(-1)^i}{F_i F_{i-1}}$$

(iii) $$\frac{1+\sqrt{5}}{2} = 1 + \sum_{i=2}^{\infty} \frac{(-1)^i}{F_i F_{i-1}}$$

[lower limits misprinted in problem statement]

Solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pa.

(i) $$F_{n+1} = \frac{F_{n+1}F_n \cdots F_2}{F_n F_{n-1} \cdots F_1} = \prod_{i=1}^{n} \frac{F_{i+1}}{F_i} = \prod_{i=1}^{n} \frac{F_i + F_{i-1}}{F_i} = \prod_{i=1}^{n} \left(1 + \frac{F_{i-1}}{F_i}\right).$$

(ii) $$\frac{F_{n+1}}{F_n} = \left(\frac{F_{n+1}}{F_n} - \frac{F_n}{F_{n-1}}\right) + \left(\frac{F_n}{F_{n-1}} - \frac{F_{n-1}}{F_{n-2}}\right) + \cdots + \left(\frac{F_3}{F_2} - \frac{F_2}{F_1}\right) + 1$$

$$= 1 + \sum_{i=2}^{n} \left(\frac{F_{i+1}}{F_i} - \frac{F_i}{F_{i-1}}\right) = 1 + \sum_{i=2}^{n} \frac{F_{i+1}F_{i-1} - F_i^2}{F_i F_{i-1}}$$

$$= 1 + \sum_{i=2}^{n} \frac{(-1)^i}{F_i F_{i-1}} ,$$

using the well-known identity,

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i .$$

(iii) In (ii) take the limit as $n \to \infty$ and recall that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$.

Also solved by Dermott A. Breault, Sylvania—ARL, Waltham, Mass.; Douglas Lind, Falls Church, Va; Charles R. Wall, Texas Christian University, Ft. Worth, Texas; and the proposer

A CORRECTED SOLUTION

B-4 *Proposed by S. L. Basin, Sylvania Electronic Systems, Mt. View, California, and Vladimir Ivanoff, San Carlos, California*

Show that $$\sum_{i=0}^{n} \binom{n}{i} F_i = F_{2n}$$

Generalize.

(Readers: Can you find the errors in the previously published solution?)

Solution by Joseph Erbacher, University of Santa Clara, Santa Clara, Calif., and J. L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania

Using the Binet formula,

$$F_{2n+j} = \frac{(a^2)^n a^j - (b^2)^n b^j}{a - b} = \frac{(1 + a)^n a^j - (1 + b)^n b^j}{a - b}$$

since

$$a^2 = a + 1, \quad b^2 = b + 1 \quad \text{when } a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2},$$

we have

$$F_{2n+j} = \frac{1}{a - b}\left[\sum_{i=0}^{n} \binom{n}{i} a^{i+j} - \sum_{i=0}^{n} \binom{n}{i} b^{i+j}\right] = \sum_{i=0}^{n} \binom{n}{i} \frac{a^{i+j} - b^{i+j}}{a - b} =$$

$$\sum_{i=0}^{n} \binom{n}{i} F_{i+j} .$$

Therefore, for arbitrary integral j,

$$F_{2n+j} = \sum_{i=0}^{n} \binom{n}{i} F_{i+j} .$$

If $j = 0$, we have the original problem. The identity also holds, with arbitrary j, for Lucas numbers since $L_n = F_{n+1} + F_{n-1}$.

CORRECTION TO VOLUME 1, NO. 1

See Vol. 1, No. 2, p. 46 for correction to last two references on page 42.