SOME NEW FIBONACCI IDENTITIES

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In this paper, some new Fibonacci and Lucas identities are generated by matrix methods.

The matrix

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

satisfies the matrix equation

$$R^3 - 2R^2 - 2R + I = 0 \quad .$$

Multiplying by R^n yields

$$R^{n+3} - 2R^{n+2} - 2R^{n+1} + R^n = 0 \tag{1}$$

It has been shown by Brennan [1] and appears in an earlier article [2] and as Elementary Problem B-16 in this quarterly that

$$R^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_nF_{n-1} & F_{n+1}^2 - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix} \quad , \tag{2}$$

where F_n is the n^{th} Fibonacci number.

By the definition of matrix addition, corresponding elements of R^{n+3}, R^{n+2}, R^{n+1} and R^n must satisfy the recursion formula given in Equation (1). That is, for example,

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0$$

and

$$F_{n+3}F_{n+4} - 2F_{n+2}F_{n+3} - 2F_{n+1}F_{n+2} + F_nF_{n+1} = 0 \quad .$$

Returning again to

$$R^3 - 2R^2 - 2R + I = 0 \quad ,$$

this equation can be rewritten as

$$(R + I)^3 = R^3 + 3R^2 + 3R + I = 5R(R + I) \quad .$$

In general, by induction, it can be shown that

$$R^p(R + I)^{2n+1} = 5^n R^{n+p}(R + I) \quad . \tag{3}$$

Equating the elements in the first row and third column of the above matrices, by means of Equation (2), we obtain

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{i+p}^2 = 5^n F_{2(n+p)+1} \quad . \tag{4}$$

It is not difficult to show that the Lucas numbers and members of the Fibonacci sequence have the relationship

$$L_n^2 - 5F_n^2 = (-1)^n 4 \quad .$$

Since also

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{i+p} = 0 \quad ,$$

we can derive the following sum of squares of Lucas numbers,

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L_{i+p}^2 = 5^{n+1} F_{2(n+p)+1} ,$$

by substitution of the preceding two identities in Equation (4).

Upon multiplying Equation (3) on the right by $(R + I)$, we obtain

$$R^p(R + 1)^{2n+2} = 5^n R^{n+p} (R + I)^2 . \tag{5}$$

Then, using the expression for R^n given in Equation (2) and the identity

$$L_k = F_{k-1} + F_{k+1} ,$$

we find that

$$(R^{n+1} + R^n)(R + I) = \begin{pmatrix} F_{2n-1} & F_{2n} & F_{2n+1} \\ 2F_{2n} & 2F_{2n+1} & 2F_{2n+2} \\ F_{2n+1} & F_{2n+2} & F_{2n+3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} L_{2n} & L_{2n+1} & L_{2n+2} \\ 2L_{2n+1} & 2L_{2n+2} & 2L_{2n+3} \\ L_{2n+2} & L_{2n+3} & L_{2n+4} \end{pmatrix} .$$

Finally, by equating the elements in the first row and third column of the matrices of Equation (5), we derive the two identities

$$\sum_{i=0}^{2n+2} \binom{2n+2}{i} F_{i+p}^2 = 5^n L_{2(n+p)}$$

and

$$\sum_{i=0}^{2n+2} \binom{2n+2}{i} L_{i+p}^2 = 5^{n+1} L_{2(n+p)} .$$

By similar steps, by equating the elements appearing in the first row and second column of the matrices of Equations (3) and (5), we can write the additional identities,

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{i-1+p} F_{i+p} = 5^n F_{2(n+p)}$$

and

$$\sum_{i=0}^{2n+2} \binom{2n+2}{i} F_{i-1+p} F_{i+p} = 5^n L_{2(n+p)+1} \quad .$$

REFERENCES

1. From the unpublished notes of Terry Brennan.
2. Marjorie Bicknell and Verner E. Hoggatt, Jr., "Fibonacci Matrices and Lambda Functions," The Fibonacci Quarterly, 1 (1963), April, pp. 47–52.

TWO CORRECTIONS, VOL. 1, NO. 4

Page 73: In proposal B-26, the last equation should read

$$B_n(x) = (x + 1) B_{n-1}(x) + b_{n-1}(x) \quad .$$

Page 74: In proposal B-27, the line for $\cos n\phi$ should read

$$\cos n\phi = P_n(x) = \sum_{j=1}^{N} A_{jn} x^{n+2-2j} \; (N = [(n+2)/2] \quad .$$