TRANSCENDENTAL NUMBERS BASED ON THE FIBONACCI SEQUENCE

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A well-known theorem due to Liouville states that if ξ is an irrational algebraic number of degree n, then the equation

$$(1) \qquad \left| \xi - \frac{p}{q} \right| < \frac{1}{q^{n+\epsilon}}$$

has only finitely many solutions for integers p, q, given any $\epsilon > 0$. Therefore, an irrational number ξ, for which

$$(2) \qquad \left| \xi - \frac{p}{q} \right| < \frac{1}{q^{t}}$$

has solutions for arbitrarily large t, must be transcendental. Numbers of this type have been called Liouville numbers.

In 1955, Roth published his celebrated improvement of Liouville's theorem, replacing "n" by "2" in equation (1). Let us call an irrational number ξ, for which

$$(3) \qquad \left| \xi - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}$$

his infinitely many solutions for some $\epsilon > 0$, a Roth number. Roth numbers are also transcendental, and they include many more numbers than the Liouville numbers.

Let b be an integer greater than 1. Then we define ξ_b to be the continued fraction

$$(4) \qquad \xi_b = \cfrac{1}{b^{F_0} + \cfrac{1}{b^{F_1} + \cfrac{1}{b^{F_2} + \cdots}}} \quad .$$

Theorem: ξ_b is a Roth number, hence ξ_b is transcendental.

Proof: From the elementary theory of continued fractions, it is well known that if p_n/q_n is the n^{th} convergent to ξ_b, then

$$\left| \xi_b - \frac{p_n}{q_n} \right| < 1/q_n q_{n+1} . \tag{5}$$

In this case, $q_0 = 1$, $q_1 = b^{F_0}$, and $q_{n+1} = b^{F_n} q_n + q_{n-1}$. We can therefore easily verify by induction that

$$q_n = \frac{b^{F_{n+1}} - 1}{b - 1} . \tag{6}$$

In particular, as $n \to \infty$ we have $q_{n+1}/q_n \to b^{F_{n+2}} / b^{F_{n+1}} = b^{F_n} \approx [(b-1)q_n]^{\phi}$ where $\phi = .618 \cdots$ is the golden ratio. Therefore for large n we have approximately

$$\left| \xi_b - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{2+\phi}}$$

and this completes the proof of the theorem.

Remarks. It can be easily shown that the set of Roth numbers is of measure zero, but it is uncountable. For example, the number $\sum_{n=1}^{\infty} b^{-c_n}$, where $\{c_n\}$ is a strictly increasing sequence of positive integers, is a Roth number if $\limsup_{n\to\infty} (c_{n+1}/c_n) > 2$, and it is a Liouville number if this lim sup is infinite. In terms of continued fractions, the number

$$\frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots$$

is a Roth number if and only if

$$\limsup_{n\to\infty} (\log a_n / \log q_n) > 0$$

where q_n are the denominators as in the proof of the above theorem.

The rapid convergence of (4) allows us to evaluate ξ_b easily with high precision, e.g.,

$$\xi_2 = .70980\ 34448\ 61291 \cdots$$
$$\xi_3 = .76859\ 75625\ 93155 \cdots$$

Reference to this article on p. 52.