ON THE USE OF FIBONACCI RECURRENCE RELATIONS IN THE DESIGN OF LONG WAVELENGTH FILTERS AND INTERFEROMETERS

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1. INTRODUCTION

The frequent occurrence of Fibonacci related numbers in art and nature has been of interest to readers of this journal since the very first issue when Basin [1] wrote such an article.

The purpose of this paper is to explain briefly the use of Horadam's generalized Fibonacci recurrence relation [2]

$$u_n = pu_{n-1} - qu_{n-2} \qquad (n \geq 2) \tag{1.1}$$

in a problem in optics. If we consider the sequence $\{u_n\}$ whose elements satisfy (1.1) with initial conditions $u_0 = 1$ and $u_1 = S_0$, then, from Horadam, we have that

$$u_n = V\alpha^n + W\beta^n$$

where

$$V = \frac{\beta - S_0}{\beta - \alpha}, \quad W = \frac{S_0 - \alpha}{\beta - \alpha},$$

in which α, β are the roots of the characteristic equation

$$x^2 - px + q = 0.$$

2. NONLINEAR RECURRENCE RELATION

To show the relationship with the optics problem, we need some more preliminary results.

Put $p = B - A$, $q = C - AB$, and the recurrence relation can be rewritten as

$$\frac{u_{n+1}}{u_{n-1}} - (B - A)\frac{u_n}{u_{n-1}} + (C - AB) = 0.$$

We then add and subtract the term

$$A\left(\frac{u_{n+1}}{u_n} + A\right)$$

to get

$$\frac{u_{n+1}}{u_{n-1}} + A\frac{u_n}{u_{n-1}} + A\frac{u_{n+1}}{u_n} + A^2 - A\frac{u_{n+1}}{u_n} - A^2 - B\frac{u_n}{u_{n-1}} - AB + C = 0,$$

which can be rewritten as

$$\left(\frac{u_{n+1}}{u_n} + A\right)\left(\frac{u_n}{u_{n-1}} + A\right) - A\left(\frac{u_{n+1}}{u_n} + A\right) - B\left(\frac{u_n}{u_{n-1}} + A\right) + C = 0.$$

From this we obtain the non-linear recurrence relation

$$R_nR_{n-1} - AR_n - BR_{n-1} + C = 0, \tag{2.1}$$

in which

(2.2)
$$R_n = \frac{u_{n+1}}{u_n} + A.$$

Thus,

(2.3)
$$R_n = \frac{(A + \alpha)V\alpha^n + (A + \beta)W\beta^n}{V\alpha^n + W\beta^n}$$

where the term S_0 in the definitions of V and W is given by

$$S_0 = R_0 - A.$$

3. THE OPTICS PROBLEM

3.1 General Remarks

In this section, we will show that Eq. (2.1) occurs in the theory of multi-element optical filters and interferometers. However, before launching into its derivation, it is necessary to acquaint the reader with the background to the problem.

Devices such as beam splitters, filters, and interferometers are common tools in all forms of experimental optics. In the fields of infrared physics and microwave engineering, the construction of such apparatus relies upon the use of wire meshes (or grids) [3]. It is possible to classify these structures into two distinct classes according to their spectral properties. These are:

a. inductive grids, made by perforating (in doubly-periodic fashion) a thin metal plate with aperatures, and
b. capacitive grids, the natural complement of inductive grids, which are composed of a periodic array of metal inclusions immersed in an insulating material.

The transmittance of inductive structures approaches zero at long wavelengths, whereas that of capacitive grids approaches unity at those wavelengths. In the far infrared and microwave regions of the electromagnetic spectrum, absorption within the metal is negligible and so we need only concern ourselves with the reflectance and transmittance of these structures.

With these prefatory remarks, let us now concern ourselves with the design of interferometers and low-frequency pass filters. These consist of a stack of many such grids separated from one another by a distance of s. In the case of the interferometer, the stack is composed of purely inductive elements, while the low-pass filter is composed of a stack of captive structures.

Each of the grids in the stack acts as a diffraction grating and gives rise to an infinite set of diffracted plane waves (orders) excited by the incident plane wave field. Interferometers and filters are operated with wavelengths in excess of the grid period (d), and so it may be deduced that only the single undispersed wave is propagating (i.e., capable of carrying energy away from the grids). All of the other orders are said to be evanescent and decay exponentially as they propagate away from a grid. Provided that the ratio s/d exceeds 0.5, the evanescent orders provide no significant mechanism for communication between the grids [4] and so we need only consider the zeroth (or undispersed) order in our derivation.

3.2 Derivation of a Non-Linear Difference Equation

Let R_0 and T_0 be the amplitude reflection and transmission coefficients of the zeroth order of one of these grids. Now consider a wave incident upon the $(n+1)$-grid structure of Figure 1. Let R_n and T_n be the amplitude reflection and transmission coefficients of this device relative to a phase origin at the center of the uppermost grid. We now regard this $(n + 1)$-grid structure as a single grid displaced by a distance s from an n-grid structure. By adopting a multiple scattering approach, we can now trace the path of a wave through this system. This is a sophistication of the ray trace mentioned by Huntley [5].

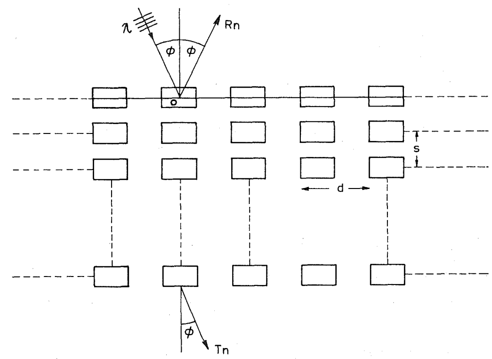

Fig. 1. Side view of the (n + 1) grid stack with a plane wave of wavelength λ indicent at an angle of φ. R_n and T_n are reflection and transmission coefficients measured relative to a phase origin at 0.

A wave of amplitude 1 incident upon the top surface is reflected with amplitude R_0 and transmitted with amplitude T_0. The transmitted component then traverses an optical path length of $s\cos\phi$ (where ϕ is the angle of incidence). Thus, the wave incident upon the n grid structure has amplitude $T_0\rho$ where

$$\rho = \exp\ (i\delta)$$

and

$$\delta = \frac{2\pi}{\lambda} s \cos\ \phi$$

for a field of wavelength λ. This wave is then transmitted and reflected by the n grid structure with the reflected amplitude being given by $T_0R_{n-1}\rho$. The reflected component then propagates toward the top surface, advancing in phase by δ, where it is partially transmitted out into free space with amplitude $T_0^2R_{n-1}\rho^2$ and partially reflected back into the cavity with amplitude $T_0R_{n-1}R_0\rho^2$; we continue this process ad infinitum and arrive at the series:

$$R_n = R_0 + R_{n-1}T_0^2\rho^2\sum_{k=0}^{\infty}(R_0R_{n-1}\rho^2)^k. \tag{3.1}$$

Since all of the reflection coefficients have magnitude less than unity, we write

$$R_n = R_0 + \frac{R_{n-1}T_0^2\rho^2}{1 - R_0R_{n-1}\rho^2}.$$

This may be reduced to the simpler form

$$R_n = \frac{R_0 - R_{n-1}\rho^2\xi^2}{1 - R_0R_{n-1}\rho^2} \tag{3.2}$$

where

$$\xi = \exp\ (i\psi_r)$$

with

$$\psi_r = \arg\ (R_0).$$

This simplification is a consequence of

a. conservation of energy,

$$|R_0|^2 + |T_0|^2 = 1$$

and

b. the phase constraint [6, 7]

$$\arg(T_0) = \arg(R_0) + \pi/2$$

appropriate to all lossless up-down symmetric structures.

Equation (3.2) is of the same form as (2.1) with

$$A = \frac{1}{R_0\rho^2}, \quad B = \frac{\xi^2}{R_0}, \quad C = \frac{1}{\rho^2};$$

constants which are only dependent upon the geometry and the reflection coefficient R_0, which may be found using a rigorous electromagnetic scattering theory. Having derived R_n, the transmittance is then

$$|T_n|^2 = 1 - |R_n|^2.$$

3.3 Interferometers

The basic component of any interferometer is an inductive element. In Figure 2 are shown transmission spectra for a typical inductive grid and its associated two-grid interferometer. The transmittance of this interferometer is given by

$$|T_1|^2 = 1/[1 + F\sin^2(\chi)]$$

where

$$F = \frac{4|R_0|^2}{(1 - |R_0|^2)^2}$$

and

$$\chi = \delta + \psi_r.$$

Clearly, it can be seen that this is a wavelength selection device with interference maxima for normally incident radiation, at

$$\lambda_{\max} = \frac{2s}{(\ell - \psi_r/\pi)} \qquad (\ell = 0, 1, 2, \ldots).$$

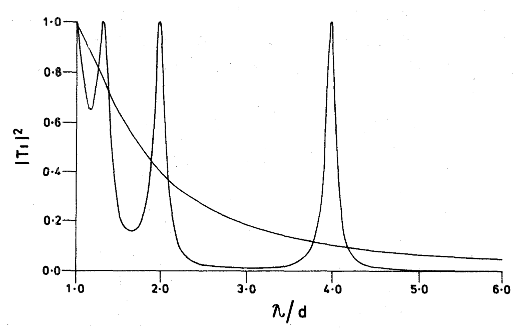

Fig. 2. Normal incidence wavelength spectra for a typical inductive grid [the curve whose decay is of the approximate form $|T_0|^2 \propto (d/\lambda)^2$] and its associated two-grid interferometer (the curve exhibiting resonant behavior at $\lambda = 2s/\ell$). For this structure, $s/d = 2.0$.

The resonance width $\Delta\lambda$ is governed by F, the finesse of the instrument, and decreases as the transmittance of a single grid decreases. This feature is illustrated in Figure 3, where a grid of substantially lower transmittance is used. Also shown on Figure 3 are spectra for three-, four-, and five-grid interferometers. For an $(n + 1)$-grid interferometer, the single transmission resonance for the two-grid device splits into n peaks, each having a significantly higher resolution factor Q,

$$Q = \frac{\lambda}{\Delta\lambda}.$$

The locations of these peaks are given by the n solutions of the equation

(3.3) $$\beta^{n+1} = \alpha^{n+1} \qquad (\beta \neq \alpha).$$

This reduces to the simpler and more explicit form,

(3.4) $$\delta + \arg(R_0) = k\pi \pm \frac{1}{2}\arccos\left[|R_0|^2 - (1 - |R_0|^2)\cos\left(\frac{2\Pi\ell}{n+1}\right)\right]$$

where $\ell = 1, 2, \ldots, n$ and k is a nonnegative integer.

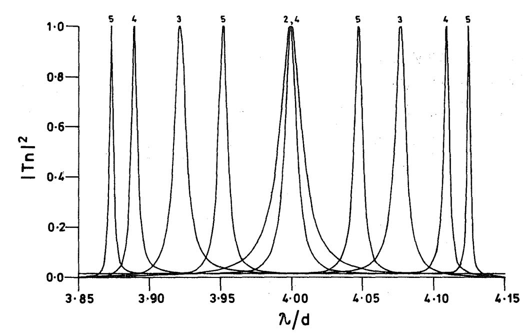

Fig. 3. Normal incidence wavelength spectra for an inductive grid stack composed of 2, 3, 4, and 5 elements (indicated by an integer above the resonance peaks). The separation of the individual elements is s/d = 2.0.

By considerably reducing the long wavelength filtering action of the inductive grids that compose the interferometer, we can obtain a broadband-pass filter. The spectrum of such a structure is illustrated in Figure 4.

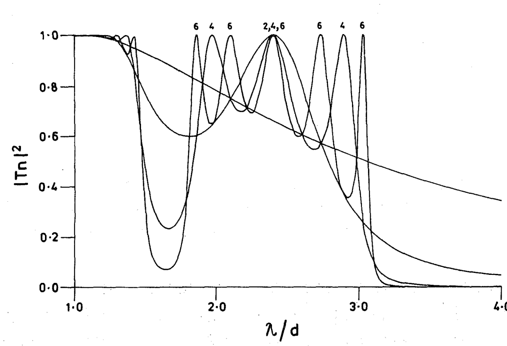

Fig. 4. Normal incidence wavelength spectra for 2-, 4-, and 6-grid band-pass filters. Here s/d = 1.2.

3.5 Low Pass Filters

The aim of such filters is to exclude any high-frequency components from the transmission spectrum. To achieve this objective, it is necessary to select a capacitive grid as the basic component of the stack. In Figure 5 we present typical spectra for four multi-element filters. Note that as the number of grids in the stack is increased, the cut-off between the transmission and rejection regions is sharpened.

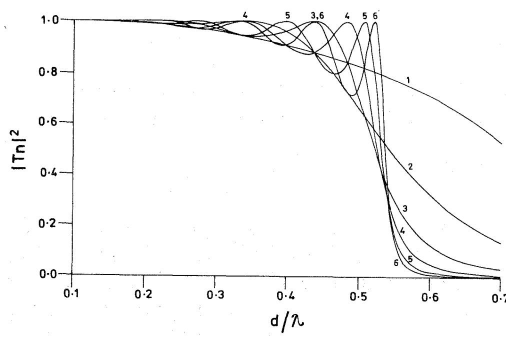

Fig. 5. Normal incidence frequency spectra of typical low-pass filters composed of up to 6 capacitive elements separated by $s/d = 1.0$.

4. CONCLUDING REMARKS

The solution of the nonlinear difference equations relying upon the use of Horadam's generalized Fibonacci recurrence relation discussed here totally circumvents the explicit and inelegant treatments of earlier, less general attempts [8]. It also facilitates the calculation of the positions of the transmission maxima [see Eq. (3.3)] for a grid stack containing an arbitrary number of elements.

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