Coefficients in the generating difference equations (4.2), as k varies, appear in Table 2 if we alternate + and - signs. Corresponding characteristic polynomials occur in [4] as proper divisors, or as products of proper divisors. Refer to Hancock [1], also.

Further, it might be noted that, if we employ the recurrence relation in (4.1) repeated.y, we may expand U_{nm} binomially as

$$U_{nm} = U_{n-t,\, m-t} + \binom{2t}{1}U_{n-t,\, m-t+1} + \binom{2t}{2}U_{n-t,\, m-t+2} + \cdots + \binom{2t}{1}U_{n-t,\, m+t+1} + U_{n-t,\, m+t} \quad (1 \le t < n,\ 1 \le t < m).$$

This is because the original recurrence relation (4.1) for U_{nm} is "binomial" (t = 1), i.e., the coefficients are 1, 2, 1.

Finally, we remark that the row elements in the first column, U_{n1}, given in (4.2), are related to the *Catalan numbers* C_n by

$$U_{n1} = (n + 1)C_n. \tag{5.5}$$

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ONE-PILE TIME AND SIZE DEPENDENT TAKE-AWAY GAMES

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1. INTRODUCTION

In a one-pile take-away game, two players alternately remove chips from a single pile of chips. Depending on the particular formulation of play, a *constraint function* specifies the number of chips which may be taken from the pile in each position. The game ends when no move is possible. In *normal* (*misère*) play, the player who makes the final move wins (loses). Necessarily, one of the players has a strategy which can force a win.

In this *Quarterly*, Whinihan [7], Schwenk [5], and Epp & Ferguson [2] have analyzed certain one-pile take-away games which can be represented by an ordered triple (n, w, f). Here $n \varepsilon Z^+ \cup \{0\}$, $w \varepsilon Z^+$, and $f: Z^+ \to Z^+$ is nondecreasing. On the initial move in the game (n, w, f), a player takes from 1 to w chips from a pile of n chips. Subsequently, if a player takes t chips from the pile, then the next player to move may take from 1 to $f(t)$ chips. In [3], the author provides an analysis of a generalization of this formulation of a one-pile take-away game so as to allow for play with two piles of chips.

The purpose of this paper is to present a formulation and an analysis of another type of one-pile take-away game. The formulation in this paper is quite dissimilar to that studied in [2], [5], and [7]. In the present formulation, the constraint function f is a function of two variables. The first variable is equal to one plus the number of moves made since the start of play. Think of this variable as representing *time*. The second variable represents the number of chips in the pile, that is, pile *size*. We shall call this formulation the *one-pile time and size dependent take-away game*. It is nicknamed *tastag*.

For example, suppose the constraint function is

$$f(t, n) = t + 1 + \left[\frac{n}{2}\right].$$

Here, $[x]$ denotes the largest integer less than or equal to x. At the start of play (time $t = 1$), suppose that the pile contains 211 chips. The first player to move may take from 1 to 107 chips. Suppose that he takes 51 chips, say, so as to leave 160 chips in the pile. Then his opponent may reply (at time $t = 2$) by taking from 1 to 83 chips. In Section 4, it will be shown that for play beginning with a pile of 211 chips, the second player to move can force a win. In Section 5, it will be shown that if the first player opens play by taking 51 chips, then the second player possesses fifteen winning replies. To force a win, the second player should take from 43 to 57 chips. If the first player opens by taking 107 chips, say, then the second player has a unique winning reply, namely, to take a single chip.

2. THE RULES OF THE GAME

Let $f: Z^+ \times Z^+ \to Z^+$. Suppose that the pile contains n chips after $t - 1$ moves have been made, $t \geq 1$. On the tth move, the player to move must take from 1 to $f(t, n)$ chips. (t, n, f) will denote the position consisting of a pile of n chips after $t - 1$ moves have been made, with play governed by the constraint function f.

In this paper we restrict ourselves to tastags for which the constraint function f satisfies the following growth condition.

CONDITION 2.1: $\forall t \geq 1$, $\forall n \geq 1$

$$f(t, n) \leq f(t, n + 1) \leq f(t, n) + 1.$$

Set $\mathcal{C} = \{f | f: Z^+ \times Z^+ \to Z^+ \text{ and } f \text{ satisfies Condition 2.1}\}$.

Define the normal outcome sets h_+ and p_+ by

$$h_+ = \{(t, n, f) | t \geq 1, n \geq 0, f \varepsilon \mathcal{C} \text{ and the } \textit{first} \text{ player to move in } (t, n, f) \text{ can force a win in } \textit{normal} \text{ play}\}$$

and

$$p_+ = \{(t, n, f) | t \geq 1, n \geq 0, f \varepsilon \mathcal{C} \text{ and the } \textit{second} \text{ player to move in } (t, n, f) \text{ can force a win in } \textit{normal} \text{ play}\}.$$

We define the misère outcome sets h_- and p_- just as we define h_+ and p_+, respectively, except that we replace "normal" by "misère" in the definitions.

For $f \varepsilon \mathcal{C}$, define $\tilde{f}: Z^+ \times Z^+ \to Z^+$ by

$$\tilde{f}(t, n) = f(t, n + 1) \quad \forall t \geq 1, \forall n \geq 1.$$

In a straightforward manner, it can be shown that $\tilde{f} \varepsilon \mathcal{C}$. It is also not difficult to verify the following:

PROPOSITION 2.1: If $t \geq 1$, $n \geq 1$, and $f \in \mathfrak{C}$, then $(t, n, f) \in h_-$ if and only if $(t, n - 1, \tilde{f}) \in h_+$.

An immediate consequence of Proposition 2.1 is the following: If we can analyze (t, n, f) for normal play for each $t \geq 1$, $n \geq 0$, and $f \in \mathfrak{C}$, then we can analyze (t, n, f) for misère play for each $t \geq 1$, $n \geq 0$, and $f \in \mathfrak{C}$.

In this paper attention is restricted to normal play. Our aim is the following:

1. Determine the outcome sets h_+ and p_+.
2. For each $(t, n, f) \in h_+$, prescribe a winning move for the player who moves next.

3. THE GAME TABLEAU

For fixed $f \in \mathfrak{C}$, to analyze all one-pile tastags (t, n, f), $t \geq 1$, $n \geq 0$, we construct a *game tableau for f*. The game tableau is an infinite array

$$\langle E_{t,r} \rangle_{t,\,r=1}^{\infty}$$

whose entries belong to the set $Z^+ \cup \{0, \infty\}$. For each $t \geq 1$, let D_t denote the tth diagonal of the tableau. That is, $D_t = \langle E_{t+1-r,\,r} \rangle_{r=1}^{t}$. For example, in the tableau in Figure 3.1, $D_8 = \langle 2, 3, 5, 0, 0, 0, 0, 0 \rangle$.

In the sequel, the following conventions are adopted:

1. $E_{t,-1} = -1$, $E_{t,0} = 0 \ \forall\, t \geq 1$.
2. $\max Z = \infty$.
3. $n + \infty = \infty \ \forall\, n \in Z^+ \cup \{0, \infty\}$.
4. The domain of f is extended from $Z^+ \times Z^+$ to $Z^+ \times (Z^+ \cup \{\infty\})$, and
$$f(t, \infty) = \infty \qquad \forall t \geq 1.$$

Construct the game tableau for f by double induction as follows:

A. The sole entry of D_1 is $E_{11} = \max\{n \mid f(1, n) \geq n\}$.

B. Suppose that the entries for diagonals $D_1, D_2, \ldots, D_{t-1}$ have been computed for some $t \geq 2$. Then compute the entries of diagonal D_t as follows:

1. $E_{t,1} = \max\{n \mid f(t, n) \geq n\}$.
2. Suppose the entries $E_{t+1-u,\,u}$, $u = 1, 2, \ldots, r - 1$, have been computed for some r, $2 \leq r \leq t$.

   a. If $E_{t-r+2,\,r-1} = 0$, put $E_{t-r+1,\,r} = 0$.

   b. If $E_{t-r+2,\,r-1} > 0$ and r is even, put
$$E_{t-r+1,\,r} = \begin{cases} 0, \text{ if } E_{t-r+2,\,r-1} + 1 \leq E_{t-r+1,\,u} \text{ for some } u,\ 1 \leq u \leq r-1. \\ E_{t-r+2,\,r-1} + 1, \text{ otherwise.} \end{cases}$$

   c. If $E_{t-r+2,\,r-1} > 0$ and r is odd, put
$$E_{t-r+1,\,r} = \begin{cases} 0, \text{ if } E_{t-r+2,\,r-1} + \max\{n \geq 1 \mid f(t-r+1, E_{t-r+2,\,r-1} + n) \geq n\} \\ \qquad\qquad \leq E_{t-r+1,\,u} \text{ for some } u,\ 1 \leq u \leq r-1. \\ E_{t-r+2,\,r-1} + \max\{n \geq 1 \mid f(t-r+1, E_{t-r+2,\,r-1} + n) \geq n\}, \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Let us illustrate this construction with an example.

EXAMPLE 3.1: Let $f: Z^+ \times Z^+ \to Z^+$ be defined as follows:

$$f(1, n) = \begin{cases} 3 & \text{for } n \leq 20, \\ n - 17 & \text{for } n \geq 21. \end{cases}$$

For $t = 2$ or 3, $f(t, n) = 5 - t + \left[\frac{n}{3}\right] \; \forall n \geq 1$.

$$f(4, n) = \begin{cases} 1 & \text{for } 1 \leq n \leq 9, \\ n - 9 & \text{for } n \geq 10. \end{cases} \quad f(5, n) \equiv 4. \quad f(6, n) = 1 + \left[\frac{n}{4}\right] \; \forall n \geq 1.$$

For $7 \leq t \leq 13$, $f(t, n) \equiv 2$. For $t \geq 14$, $f(t, n) = n \; \forall n \geq 1$. Condition 2.1 is satisfied by f. The complete game tableau for f is given in Figure 3.1.

t \ r	1	2	3	4	5	6	7	8	9	10	11	12	...
1	3	5	0	0	14	0	19	0	∞	0	0	0	...
2	4	0	0	11	0	16	0	20	0	0	∞	0	...
3	3	0	10	0	15	0	19	0	0	∞	0	0	...
4	1	5	0	8	0	11	0	0	∞	0	0	0	...
5	4	0	7	0	10	0	0	14	0	∞	0	0	...
6	1	3	5	6	9	0	13	0	∞	0	0	0	...
7	2	3	5	6	8	9	11	∞	0	0	0	...	
8	2	3	5	6	8	9	∞	0	0	0	...		
9	2	3	5	6	8	∞	0	0	0	...			
10	2	3	5	6	∞	0	0	0	...				
11	2	3	5	∞	0	0	0	...					
12	2	3	∞	0	0	0	...						
13	2	∞	0	0	0	...							
14	∞	0	0	0	...								
15	∞	0	0	...									
16	∞	0		...									
⋮	⋮	⋮	⋮										

Fig. 3.1. The game tableau for Example 3.1

For a large class of constraint functions in $\mathcal{C}$, the corresponding game tableaux have no zero entries. For any such game tableau, the entries of each row (column) form a strictly increasing (nondecreasing) sequence of positive integers. The tastags generated by such constraint functions will be called *escalation tastags*. Set

$$\mathcal{E} = \{f \in \mathcal{C} \mid \text{the game tableau of } f \text{ has no zero entries}\}.$$

EXAMPLE 3.2: Consider the constraint function $f(t) = t + 1 + [n/2]$ mentioned in Section 1. For $t \geq 1$ and $r \geq 1$, it can be shown that

$$E_{t,r} = \begin{cases} [2(r + t) - 3]2^{(r+1)/2} - 2(t - 2) & \text{if } r \text{ is odd,} \\ [2(r + t) - 3]2^{r/2} - 2t + 3 & \text{if } r \text{ is even.} \end{cases}$$

$f \in \mathcal{E}$. A portion of the tableau of f is shown in Figure 3.2.

EXAMPLE 3.3: On page 124 of [6], Silverman introduces a game called *Triskidekaphilia Escalation*. It was the challenge of this game for an arbitrary pile size $n \geq 0$ that motivated the present study of one-pile tastags. This game is equivalent to the one-pile tastag $(1, n, f)$, where $f(t, n) = t + 1$. $f \in \mathcal{E}$. For $t \geq 1$ and $r \geq 1$, it can be shown that

$$E_{t,r} = \begin{cases} \left(\frac{r+1}{2}\right)^2 + (t+1)\left(\frac{r+1}{2}\right) - 1 & \text{if } r \text{ is odd,} \\ \left(\frac{r}{2}\right)^2 + (t+2)\left(\frac{r}{2}\right) & \text{if } r \text{ is even.} \end{cases}$$

A portion of the game tableau of f is shown in Figure 3.3.

$t \backslash r$	1	2	3	4	5	6	7	8
1	4	7	22	29	74	89	210	241
2	6	9	28	35	88	103	240	271
3	8	11	34	41	102	117	270	301
4	10	13	40	47	116	131	300	331
5	12	15	46	53	130	145	330	361
6	14	17	52	59	144	159	360	391
7	16	19	58	65	158	173	390	421
8	18	21	64	71	172	187	420	451

Fig. 3.2. A portion of the game tableau for Examle 3.2

$t \backslash r$	1	2	3	4	5	6	7	8	9	10
1	2	4	7	10	14	18	23	28	34	40
2	3	5	9	12	17	21	27	32	39	45
3	4	6	11	14	20	24	31	36	44	50
4	5	7	13	16	23	27	35	40	49	55
5	6	8	15	18	26	30	39	44	54	60
6	7	9	17	20	29	33	43	48	59	65
7	8	10	19	22	32	36	47	52	64	70
8	9	11	21	24	35	39	51	56	69	75
9	10	12	23	26	38	42	55	60	74	80
10	11	13	25	28	41	45	59	64	79	85

Fig. 3.3 A portion of the game tableau for Example 3.3

4. DETERMINING THE NORMAL OUTCOME SETS

From the game tableau of f, $f \in \mathcal{C}$, the following theorem reveals the outcome set to which any tastag (t, n, f) belongs.

THEOREM 4.1: If $t \geq 1$, $n \geq 1$, and $f \in \mathcal{C}$, then $(t, n, f) \in h_+$ if and only if $\min\{r | E_{t,r} \geq n\}$ is odd.

As an illustration, return to Example 3.1. Is $(1, 22, f)$ a first-player win? Here

$$\min\{r | E_{1,r} \geq 22\} = 9,$$

which is odd. Thus, the first player to move in $(1, 22, f)$ can force a win.

How about the position $(5, 11, f)$? Here

$$\min\{r | E_{5,r} \geq 11\} = 8,$$

which is even. Thus, the second player to move in $(5, 11, f)$ can force a win.

As a final example, return to the tastag $(1, 211, f)$ mentioned in Section 1: $f(t, n) = t + 1 + [n/2]$. A portion of the game tableau for f is shown in Figure 3.2. We observe that $\min\{r | E_{1,r} \geq 211\} = 8$, which is even. As asserted in Section 1, $(1, 211, f)$ is a second-player win.

In the author's doctoral dissertation [4], it is shown that if $f \in \mathcal{C}$, then $\min\{r|E_{1,r} \geq n\}$ is, in fact, the normal *remoteness* number of (t, n, f). Moveover, if $f \in \mathcal{E}$, then $\min\{r|E_{1,r} \geq n\}$ is also the normal *suspense* number of (t, n, f).*

5. AN OPTIMAL STRATEGY

The proof of Theorem 4.1 will be *constructive*. Suppose that $(t, n, f) \in h_+$. Set $\beta(t, n, f) = \min\{r|E_{t,r} \geq n\}$. We prescribe the following *winning move*:

1. Take $n - E_{t+1,\, \beta(t,n,f)-1}$ chips if $n > E_{t+1,\, \beta(t,n,f)-1}$.
2. Take a single chip if $n \leq E_{t+1,\, \beta(t,n,f)-1}$.

As an illustration, return again to Example 3.1.

First consider the position $(3, 19, f)$. $\beta(3, 19, f) = 7$, so $(3, 19, f) \in h_+$. $19 > 11 = E_{4,6}$. The player whose turn it is to move should take $19 - 11 = 8$ chips. Since $f(3, 19) = 8$, seven other moves are also possible. Observe that each of the seven other moves is "bad," since $\beta(4, 19 - u, f) = 9\ \forall u,\ 1 \leq u \leq 7$.

Next consider the position $(4, 13, f)$. $\beta(4, 13, f) = 9$, so $(4, 13, f) \in h_+$. $13 \leq 14 = E_{5,8}$. The first player to move can make a winning move by taking a single chip. $f(4, 13) = 4$. Note that taking 2 chips is also a winning move. However, taking either 3 or 4 chips is a losing move.

Let u denote the move in which u chips are taken from the pile. The *set of winning moves* from the position (t, n, f) is

$$\{u|1 \leq u \leq f(t, n) \wedge n, \text{ and } (t + 1, n - u, f) \in p_+\}$$

$$= \{u|1 \leq u \leq f(t, n) \wedge n, \text{ and } \beta(t + 1, n - u, f) \text{ is even}\}.$$

When this set is nonempty, Condition 2.1 and a short argument assures us that it is a set of *consecutive integers*.

Return to the tastag discussed in Section 1. From Figure 3.2 we observe that $\beta(2, 160, f) = 7$, so $(2, 160, f) \in h_+$. The set of winning moves from $(2, 160, f)$ is

$$\{u|1 \leq u \leq 83, \text{ and } \beta(3, 160 - u, f) = 6\} = \{43, 44, \ldots, 57\}.$$

Next note that $\beta(2, 104, f) = 7$. The set of winning moves from $(2, 104, f)$ is

$$\{u|1 \leq u \leq 55, \text{ and } \beta(3, 104 - u, f) = 6\} = \{1\}.$$

6. THE PROOF OF THEOREM 4.1

Our proof of Theorem 4.1 takes the usual approach. Pick any $f \in \mathcal{C}$. To show that a set A satisfies

$$A = \{(t, n, f)|t \geq 1, n \geq 0\} \cap h_+,$$

it suffices to show each of the following:

a. No terminal position is in A.
b. For each position in A, there exists a move to a position not in A.
c. For each position not in A, every move results in a position in A.

Before proving Theorem 4.1, we introduce some notation and prove two lemmas.

For each $t \geq 1$, $n \geq 0$, *define*

$$\alpha(t, n, f) = \max[\{0\} \cup \{r|0 < E_{t,r} < n\}],$$
$$\beta(t, n, f) = \min\{r|E_{t,r} \geq n\}, \text{ and}$$
$$\gamma(t, n, f) = \max[\{0\} \cup \{r|r \text{ is even}, E_{t+1,r} > 0, r < \beta(t, n, t)\}].$$

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*Chapter 14 of [1] is a good reference for the reader who is not familiar with the concepts of remoteness and suspense numbers.

Since $f(t, n) \geq 1 \; \forall \; t \geq 1, \; \forall \, n \geq 1$, it can be shown that $\beta(t, n, f) < \infty$ [in fact, $\beta(t, n, f) \leq n$] $\forall \; t \geq 1, \; \forall \, n \geq 0$. *Define* the set of "followers" of position (t, n, f) to be

$$F(t, n, f) = \{(t + 1, n - u, f) \,|\, 1 \leq u \leq f(t, n) \wedge n\}.$$

For each $\ell \geq 0$, *define* the set

$$A_\ell = \{(t, n, f) \,|\, t \geq 1, n \geq 0, \beta(t, n, f) \text{ is odd}\}.$$

Theorem 4.1 asserts that

$$\{(t, n, f) \,|\, t \geq 1, n \geq 0\} \cap h_+ = \bigcup_{r=0}^{\infty} A_{2r+1}.$$

Demanding that f satisfies Condition 2.1 forces the game tableau of f to possess two nice properties. Lemma 6.1 reveals the two properties.

LEMMA 6.1: Suppose $f \in \mathcal{C}$, $t \geq 1$, and $r \geq 0$.

a. If $0 < E_{t,\,2r+1} < n$, then $n - f(t, n) > E_{t+1,\,2r}$.
b. If $0 < n \leq E_{t,\,2r+1}$, then $n - f(t, n) \leq E_{t+1,\,2r}$.

PROOF: a. By the manner in which the tableau is constructed,

$$E_{t,\,2r+1} > 0 \Rightarrow E_{t,\,2r+1} = E_{t+1,\,2r} + \delta,$$

where $\delta = \max\{n' \,|\, f(t, E_{t+1,\,2r} + n') \geq n'\}$. Observe that

(1) $$f(t, E_{t+1,\,2r} + \delta + 1) < \delta + 1.$$

Since $n > E_{t,\,2r+1}$, we have $n - E_{t+1,\,2r} - \delta - 1 \geq 0$. Thus,

(2) $$\begin{aligned} f(t, n) &= f[t, (E_{t+1,\,2r} + \delta + 1) + (n - E_{t+1,\,2r} - \delta - 1)] \\ &\leq f(t, E_{t+1,\,2r} + \delta + 1) + (n - E_{t+1,\,2r} - \delta - 1) \end{aligned}$$

by Condition 2.1. (1) and (2) yield

$$f(t, n) < (\delta + 1) + (n - E_{t+1,\,2r} - \delta - 1) = n - E_{t+1,\,2r}.$$

Thus, $n - f(t, n) > E_{t+1,\,2r}$.

b. Since $E_{t,\,2r+1} > 0$, we have $E_{t,\,2r+1} = E_{t+1,\,2r} + \delta$, where δ is as in the proof of part (a) of the Lemma. If $n - 1 \leq E_{t+1,\,2r}$, then the assertion in part (b) of the Lemma is trivial. So suppose $n > E_{t+1,\,2r} + 1$. Then $1 < n - E_{t+1,\,2r} \leq \delta$, and so

$$f(t, n) = f[t, E_{t+1,\,2r} + (n - E_{t+1,\,2r})] \geq n - E_{t+1,\,2r}. \qquad \text{Q.E.D.}$$

The second lemma we shall need is the following.

LEMMA 6.2: Suppose $f \in \mathcal{C}$, $t \geq 1$, $r \geq 1$, and $E_{t,u} < \infty$ for each u, $1 \leq u \leq 2r$. If $E_{t+1,\,2r} > 0$, then $E_{t,\,2r+1} > 0$.

PROOF: Suppose $E_{t,u} < \infty$ for each u, $1 \leq u \leq 2r$, and suppose $E_{t+1,\,2r} > 0$. Then $E_{t,\,2r+1} = 0$ if and only if

$$\exists u, \; 1 \leq u \leq 2r, \; \ni E_{t,u} \geq E_{t+1,\,2r} + \delta,$$

where $\delta = \max\{n \,|\, f(t, E_{t+1,\,2r} + n) \geq n\}$. *Assume* that there exists such an integer u. We consider two cases.

Case 1. u is even. Here $\exists r'$, $1 \leq r' \leq r$, $\ni u = 2r'$. Since

$$E_{t+1,\,2r} > 0, \; E_{t+1,\,2r'-1} < E_{t+1,\,2r}.$$

Thus

$$E_{t+1,\,2r} + \delta \geq E_{t+1,\,2r} + 1 > E_{t+1,\,2r'-1} + 1 = E_{t,\,2r'} = E_{t,u},$$

a contradiction.

Case 2. u is odd. Here $\exists r'$, $0 \le r' < r$, $\ni u = 2r' + 1$. Let

$$\delta' = \max\{n \mid f(t, E_{t+1,\, 2r'} + n) \ge n\},$$

so $E_{t,\, 2r'+1} = E_{t+1,\, 2r'} + \delta'$. Then

$$\begin{aligned} &f[t, E_{t+1,\, 2r} + (E_{t,\, 2r'+1} - E_{t+1,\, 2r} + 1)] \\ &\quad = f[t, E_{t+1,\, 2r} + (E_{t+1,\, 2r'} + \delta' - E_{t+1,\, 2r} + 1)] \\ &\quad = f(t, E_{t+1,\, 2r'} + \delta' + 1) \ge f(t, E_{t+1,\, 2r'} + \delta') \quad \text{by Condition 2.1} \\ &\quad \ge \delta' \quad \text{by the definition of } \delta' \\ &\quad = E_{t,\, 2r'+1} - E_{t+1,\, 2r'} \quad \text{since } E_{t,\, 2r'+1} = E_{t+1,\, 2r'} + \delta' \\ &\quad \ge E_{t,\, 2r'+1} - E_{t+1,\, 2r} + 1 \quad \text{since } E_{t+1,\, 2r} > 0 \Rightarrow E_{t+1,\, 2r} > E_{t+1,\, 2r'}. \end{aligned}$$

Thus, $\delta \ge E_{t,\, 2r'+1} - E_{t+1,\, 2r} + 1$. Consequently,

$$E_{t+1,\, 2r} + \delta \ge E_{t+1,\, 2r} + (E_{t,\, 2r'+1} - E_{t+1,\, 2r} + 1) > E_{t,\, 2r'+1} = E_{t,u},$$

a contradiction.

In both Case 1 and Case 2, a contradiction has been observed. Thus, it must be that $E_{t,\, 2r+1} > 0$. Q.E.D.

PROOF OF THEOREM 4.1: Consider the set

$$A = \bigcup_{r=0}^{\infty} A_{2r+1}.$$

To prove Theorem 4.1, it suffices to establish statements (a), (b), and (c) in the first paragraph of this section. Figure 6.1 is intended as a guide.

$$\begin{array}{cccccccc} & \ldots\ E_{t,\gamma+1}\ \ldots & & E_{t,\alpha} & \ldots & E_{t,\beta}\ \ldots & \\ \ldots\ E_{t+1,\gamma}\ \ldots & & E_{t+1,\alpha-1} & \ldots & E_{t+1,\beta-1}\ \ldots & & \end{array}$$

Fig. 6.1. A portion of the game tableau for f

a. The set of terminal positions is $\{(t, 0, f) \mid t \ge 1\}$.

$$\beta(t, 0, f) = 0\ \forall\ t \ge 1, \text{ since } E_{t,0} = 0\ \forall\ t \ge 1.$$

Thus, $\underline{\{\text{terminal positions}\} \cap A = \emptyset}$. Statement (a) holds.

b. Suppose $\underline{(t, n, f) \in A}$. Then $\beta(t, n, f)$ is odd. Let $\alpha = \alpha(t, n, f)$ and $\beta = \beta(t, n, f)$. There are two cases to consider.

Case b.1. $n > E_{t+1,\, \beta-1}$. Since $0 < n \le E_{t,\beta}$, part (b) of Lemma 6.1 indicates that $n - f(t, n) \le E_{t+1,\, \beta-1}$. Thus, in position (t, n, f), a player may take

$$\underline{n - E_{t+1,\, \beta-1}}$$

chips to leave the position $(t + 1, E_{t+1,\beta-1}, f)$. $\beta(t + 1, E_{t+1,\beta-1}, f) = \beta - 1$ is even, so

$$\underline{(t + 1, E_{t+1,\, \beta-1}, f) \notin A}.$$

Case b.2. $n \le E_{t+1,\, \beta-1}$. Taking a single chip leaves the position

$$(t + 1, n - 1, f).$$

Let $\beta' = \beta(t + 1, n - 1, f)$. Since $E_{t+1,\, \beta-1} > n - 1$, we have $\beta' \le \beta - 1$, and so $\beta' + 1 \le \beta$.

Assume that $(t+1, n-1, f) \in A$. Then β' is odd. Set $\tilde{E}_{t,\beta'+1} = E_{t+1,\beta'} + 1$. Since $E_{t+1,\beta'} \geq n-1$, we have

(3) $$\tilde{E}_{t,\beta'+1} \geq n.$$

Consequently,

(4) $$\tilde{E}_{t,\beta'+1} > E_{t,\alpha}.$$

By the maximality of α, the minimality of β, and (4), we conclude that $E_{t,\beta'+1} > 0$ (and, of course, $E_{t,\beta'+1} = \tilde{E}_{t,\beta'+1}$). But $\beta'+1$ is even, β is odd, and $\beta'+1 \leq \beta$. Hence, we also have $\beta'+1 < \beta$. $\beta'+1 < \beta$ and (3) contradict the minimality of β. We conclude that $(t+1, n-1, f) \notin A$.

We have shown that, in both Case b.1 and Case b.2, statement (b) holds.

c. Suppose $\underline{(t, n, f) \notin A}$. If $n = 0$, statement (c) is vacuous. So assume $n > 0$. Observe that β is even and that $\beta > 0$. Let $\gamma = \gamma(t, n, f)$. If $\gamma = 0$, then $E_{t,\gamma+1} > 0$. If $\gamma > 0$, then γ even and $E_{t+1,\gamma} > 0$ imply that $E_{t,\gamma+1} > 0$ by Lemma 6.2. Thus, in either case, $E_{t,\gamma+1} > 0$. So $\underline{\gamma + 1 \leq \alpha}$ by the maximality of α, the minimality of β, and the fact that $\alpha + 1 < \beta$.

Now $0 < E_{t,\gamma+1} \leq E_{t,\alpha} < n$ and γ even imply that

(5) $$n - f(t, n) > E_{t+1,\gamma}$$

by (a) of Lemma 6.1. Since $n \leq E_{t,\beta} = E_{t+1,\beta-1} + 1$, $n - 1 \leq E_{t+1,\beta-1}$. Combine this with (5) to get

$$\underline{E_{t+1,\gamma} < n - u \leq E_{t+1,\beta-1} \ \forall u \ni 1 \leq u \leq f(t, n).}$$

Thus, $\beta(t+1, n-u, f)$ is odd $\forall u \ni 1 \leq u \leq f(t, n)$. We have shown that

$$\underline{F(t, n, f) \subseteq A},$$

which verifies statement (c). Q.E.D.

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Horadam [2] defined and studied in detail the generalized Fibonacci sequence defined by