GENERALIZED FIBONACCI NUMBERS BY MATRIX METHODS

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In [7], Silvester shows that a number of the properties of the Fibonacci sequence can be derived from a matrix representation. In so doing, he shows that if $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ then

$$A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix}, \tag{1}$$

where u_k represents the kth Fibonacci number. This is a special case of a more general phenomenon. Suppose the $(n+k)$th term of a sequence is defined recursively as a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1} \tag{2}$$

(c_0, ..., c_{k-1} are constants). Given values for the first k terms, a_0, a_1, ..., a_{k-1}, (2) uniquely determines a sequence $\{a_n\}$. In this context, the Fibonacci sequence $\{u_n\}$ may be viewed as the solution to

$$a_{n+2} = a_n + a_{n+1}$$

which has initial terms $u_0 = 0$ and $u_1 = 1$.

Difference equations of the form (2) are expressible in a matrix form analogous to (1). This formulation is unfortunately absent in some general works on difference equations (e.g. [2], [4]), although it has been used extensively by Bernstein (e.g. [1]) and Shannon (e.g. [6]). Define the matrix A by

$$A = \begin{bmatrix} 0 & 1 & 0 \ldots 0 & 0 \\ 0 & 0 & 1 \ldots 0 & 0 \\ 0 & 0 & 0 \ldots 0 & 0 \\ \vdots & \vdots & \vdots \quad\; \vdots & \vdots \\ 0 & 0 & 0 \ldots 0 & 1 \\ c_0 & c_1 & c_2 \quad c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then, by an inductive argument, we reach the generalization of (1):

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}. \tag{3}$$

Just as Silvester derived many interesting properties of the Fibonacci numbers from a matrix representation, it also is possible to learn a good deal about $\{a_n\}$ from (3). We will confine ourselves to deriving a general formula for a_n as a function of n valid for a large class of equations (2). The reader is invited to generalize our results and explore further consequences of (3).

Following Shannon [5], we define a generalized Fibonacci sequence as a solution to (2) with the initial terms $[a_0, \ldots, a_{k-1}] = [0, 0, \ldots, 0, 1]$. Equation (3) then becomes

$$\begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k} \end{bmatrix} = A^n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

More specifically, a formula for a_n is given by

$$(4)\qquad a_n = [1 \quad 0 \quad 0 \ldots 0]A^n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

When A can be brought to diagonal form, (4) is easily evaluated to provide the desired formula for a_n.

As many readers have doubtless recognized, A is the companion matrix for the polynomial

$$(5)\qquad p(t) = t^k - c_{k-1}t^{k-1} - c_{k-2}t^{k-2} - \cdots - c_0.$$

In consequence, $p(t)$ is both the characteristic and minimal polynomial for A, and A can be diagonalized precisely when p has k distinct roots. In this case we have

$$(6)\qquad p(t) = (t - r_1)(t - r_2) \ldots (t - r_k)$$

and the numbers $r_1, r_2, \ldots, r_k$ are the eigenvalues of A.

To determine an eigenvector for A corresponding to the eigenvalue r_i we consider the system

$$(7)\qquad (A - r_iI)X = 0.$$

As there are k eigenvalues, each must have geometric multiplicity one, and so the rank of $(A - r_iI)$ is $k - 1$. The general solution to (7) is readily preceived as

$$X = x_1 \begin{bmatrix} 1 \\ r_i \\ r_i^2 \\ \vdots \\ r_i^{k-1} \end{bmatrix}$$

where x_1 may be any scalar. For convenience, we take $x_1 = 1$.

Following the conventional procedure for diagonalizing A, we invoke the factorization

$$A = SDS^{-1},$$

where S is a matrix with eigenvectors of A for columns and D is a diagonal matrix. Interestingly, the previous discussion shows that for a polynomial p with distinct roots $r_1, r_2, \ldots, r_k$, the companion matrix A can be diagonalized by choosing S to be the Vandermonde array

$$V(r_1, r_2, \ldots, r_k) = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ r_1 & r_2 & r_3 & \ldots & r_k \\ r_1^2 & r_2^2 & r_3^2 & \ldots & r_k^2 \\ & & \vdots & & \\ r_1^{k-1} & r_2^{k-1} & r_3^{k-1} & \ldots & r_k^{k-1} \end{bmatrix}.$$

Related results have been previously discussed in Jarden [3].

To make use of the diagonal form, we substitute for A in (4) and derive the following:

$$a_n = [1 \quad 0 \quad 0 \ \ldots \ 0]V(r_1, r_2, \ldots, r_k)D^nV^{-1}(r_1, r_2, \ldots, r_k)\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Noting that the product of the first three matrices at right is $[r_1^n \ r_2^n \ \ldots \ r_k^n]$, we represent the product of the remaining matrices by

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$$

and a much simpler formula for a_n results:

$$a_n = \sum_{i=1}^{k} r_i^n y_i. \tag{8}$$

Now, to determine the values $y_1, \ldots, y_k$, we solve

$$V(r_1, r_2, \ldots, r_k)\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

By Cramer's rule, y_m is given by the ratio of two determinants. In the numerator, after expanding by minors in column m, the result is

$$(-1)^{m+k}\det V(r_1, \ldots, r_{m-1}, r_{m+1}, \ldots, r_k),$$

while the denominator is $\det V(r_1, \ldots, r_k)$. Thus, the ratio simplifies to

$$y_m = \frac{(-1)^{m+k}}{(-1)^{k-m}\prod_{i \neq m}(r_m - r_i)}.$$

The final form of the formula is derived by utilizing the notation of (6) and recognizing the last product above as $p'(r_m)$. Substitution in (8), and elimination of the factors of (-1) complete the computations and produce a simple formula for a_n:

$$a_n = \sum_{i=1}^{k} \frac{r_i^n}{p'(r_i)} \tag{9}$$

We conclude with a few examples and comments that pertain to the case $k = 2$. Taking $c_0 = c_1 = 1$, the sequence $\{a_n\}$ is the Fibonacci sequence. Here

$$p(t) = t^2 - t - 1 = \left(t - \frac{1+\sqrt{5}}{2}\right)\left(t - \frac{1-\sqrt{5}}{2}\right)$$

and $p'(t) = 2t - 1$. By using (9), we derive the familiar formula:

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}} + \frac{\left(\frac{1-\sqrt{5}}{2}\right)^n}{-\sqrt{5}}.$$

Consider next the case $c_0 = c_1 = 1/2$, in which each term in the sequence is the average of the two preceding terms. Now,

$$p(t) = t^2 - \frac{1}{2}t - \frac{1}{2} = (t - 1)\left(t + \frac{1}{2}\right).$$

This time, (9) leads to

$$a_n = \frac{1}{3}\left[2 + \left(-\frac{1}{2}\right)^{n-1}\right].$$

More generally for $k = 2$, the discriminant of $p(t)$ will be $D = c_1^2 + 4c_0$ and (9) produces the formula

$$a_n = \frac{(c_1 + \sqrt{D})^n - (c_1 - \sqrt{D})^n}{2^n\sqrt{D}}.$$

If D is negative, we may express the complex number $c_1 + \sqrt{D}$ in polar form as

$$R(\cos\theta + i\sin\theta).$$

Then the formula for a_n simplifies to

$$a_n = \left(\frac{R}{2}\right)^{n-1}\frac{\sin n\theta}{\sin\theta}.$$

Thus, for example, with $c_1 = c_0 = -1$, we obtain

$$a_n = (-1)^{n-1}\frac{2}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right).$$

This sequence $\{a_n\}$ is periodic, repeating 0, 1, -1, as may be verified inductively from the original difference equation

$$a_{n+2} = -a_n - a_{n+1};\ a_0 = 0;\ a_1 = 1.$$

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