B-271(b) (12, 1974): If k is even, then $\ell_k - 2c^k$ divides

$$h_{(n+2)k} - 2h_{(n+1)k}c^k + h_{nk}c^k.$$

(This generalization was suggested by the referee.)

B-275 (13, 1975): $h_{mn} = \ell_m h_{m(n-1)} - (-c)^m h_{m(n-2)}$.

B-277 (13, 1975): $\ell_{2n(2k+1)} \equiv c^{2nk}\ell_{2n} \pmod{f_{2n}}$.

B-282 (13, 1975): If $c = d^2$ $(d > 0)$, then $2d\ell_n\ell_{n+1}$, $|\ell_{n+1}^2 - c\ell_n^2|$, and $c\ell_{2n} + \ell_{2n+2}$ are the lengths of a right-angled triangle.

B-294 (13, 1975): $h_n\ell_k + h_k\ell_n = 2h_{n+k} + q(-c)^k\ell_{n-k}$.

B-298 (14, 1976): $(b^2 + 4c)h_{2n+3}h_{2n-3} = p^2\ell_{4n} + 2cpq\ell_{4n-1} + q^2c^2\ell_{4n-2} + ec^{2n-3}\ell_6$, where $e = p^2 - bpq - cq^2 = \ell m$.

B-323 (15, 1977): $h_{n+t}^2 - (-c)^t h_n^2 = f_t(ph_{2n+t} + cqh_{2n+t-1})$.

B-342 (15, 1977): $2c^3\ell_{n-1}^3 + b^3\ell_n^3 + 6c\ell_{n+1}^2\ell_{n-1} = (\ell_{n+1} + c\ell_{n-1})^3$.

B-343 (15, 1977): $\displaystyle\sum_{k=1}^{n}[cf_{2k-1}f_{2(n-k)+1} - f_{2k}f_{2(n-k+1)}] = \frac{1}{b^2 + 4c}\left(\frac{4c^2}{b}f_{2n} - bn\ell_{2n+1}\right)$.

B-354 (16, 1978): $h_{n+k}^3 - \ell_k^3 h_n^3 + (-c)^k h_{n-k}[c^{2k}h_{n-k}^2 + 3h_{n+k}h_n\ell_k] = 0$.

B-355 (16, 1978): $h_{n+k}^3 - \ell_{3k}h_n^3 + (-c)^{3k}h_{n-k}^3 = 3e(-c)^n h_n f_k f_{2k}$.

B-379 (17, 1979): $f_{2n} \equiv nb(-c)^{n-1}$ [mod $(b^2 + 4c)$] for $n = 1, 2, \ldots$.

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A VARIANT OF THE FIBONACCI POLYNOMIALS WHICH ARISES IN THE GAMBLER'S RUIN PROBLEM

MARK E. KIDWELL and CLIFFORD M. HURVICH
Amherst College, Amherst, MA 01002
(Submitted September 1980)

In examining the gambler's ruin problem (a simple case of random walk) with a finite number of possible states, we were led to consider a sequence of linear recurrence relations that describe the number of ways to reach a given state. These recurrence relations have a sequence of polynomials as their auxiliary equations. These polynomials were unknown to us, but proved exceptionally rich in identities. We gradually noticed that these identities were analogous to well-known identities satisfied by the Fibonacci numbers. A check of back issues of *The Fibonacci Quarterly* then revealed that our sequence of polynomials differed only in sign from the Fibonacci polynomials studied in [1], [5], and several other papers.

In this paper we show, using graph theory and linear algebra, how the gambler's ruin problem gives rise to our sequence of polynomials. We then compare our polynomials to the Fibonacci polynomials and explain why the two sequences satisfy analogous identities. Finally, we use the Pascal arrays introduced in our analysis of gambler's ruin to give a novel proof of the divisibility properties of our sequence.

The Fibonacci numbers are defined recursively by

$$F_0 = 0,\ F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2},\ n \geq 2.$$

Likewise, the Fibonacci polynomials are defined by

$$F_0(x) = 0,\ F_1(x) = 1, \text{ and } F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ n \geq 2$$

(see [1, p. 407]).

1. GAMBLER'S RUIN AND PASCAL ARRAYS

A gambler whose initial capital is j dollars enters a game consisting of a sequence of discrete rounds. Each round is either won or lost. If the gambler wins a round, he is awarded one dollar; if he loses the round, he must forfeit one dollar. The game continues until either:

1. His capital reaches 0 for the first time. (Ruin.)
2. His capital reaches $b > 1$ for the first time. (Victory.)

Zero is called the *lower barrier* and b the *upper barrier*. Since the game ends as soon as either barrier is reached, these barriers are *absorbing* [3, p. 342].

We are interested in the number of ways the gambler's capital can reach i dollars, $0 < i < b$, in n rounds. Since he gains or loses one dollar in each round, this number equals the sum of the number of ways his capital can reach $i - 1$ or $i + 1$ dollars in $n - 1$ rounds, provided that $i - 1$ and $i + 1$ do not lie on the barriers. These numbers thus satisfy a recursive relation similar to that of the binomial coefficients in Pascal's triangle, except for the interference of the barriers.

Following Feller [3, Ch. 3], we use a "left-to-right" format for our truncated Pascal triangle rather than a "top-to-bottom" format. Thus in Diagram 1, we plot the numbers we have been describing on integer lattice points (n, i) with $b = 5$. We make the initial capital three dollars.

Capital	0	1	2	3	4	5	6	7	
5									Barriers
4	0	1	0	2	0	5	0	13	
3	1	0	2	0	5	0	13	0	
2	0	1	0	3	0	8	0	21	
1	0	0	1	0	3	0	8	0	
0									Barriers

Number of Rounds

Diagram 1

The appearance of the Fibonacci numbers F_n and F_{n+1} in the nth column is an accidental consequence of the selection of $b = 5$. In speaking of the point (n, i), we are using the "column first, row second" convention that is standard for coordinate systems, not the "row first, column second" convention of matrix theory.

It will be useful later to employ this rectangular lattice with more general initial values (the values in the 0th column). Given an integer $b > 1$ and a vector $\vec{X}_0 \in \mathbb{C}^{b-1}$, we define the *Pascal Array* P.A. $(b, \vec{X}_0)$, of height b and initial vector X_0, to be the (complex) array whose (n, i)-entry, for $n \geq 0$ and $0 < i < b$, is

$$(1)\qquad F(n, b, \vec{X}_0, i) = \begin{cases} \vec{X}_0 \cdot \vec{e}_i \text{ if } n = 0 \\ F(n-1, b, \vec{X}_0, i-1) + F(n-1, b, \vec{X}_0, i+1) \\ \text{if } n > 0 \text{ and } 1 < i < b - 1 \\ F(n-1, b, \vec{X}_0, 2) \text{ if } n > 0 \text{ and } i = 1 \\ F(n-1, b, \vec{X}_0, b-2) \text{ if } n > 0 \text{ and } i = b - 1. \end{cases}$$

($\vec{e}_i$ is the ith standard unit vector in $\mathbb{C}^{b-1}$.) Thus, in gambler's ruin, we are dealing with P.A. $(b, \vec{e}_j)$, $1 \le j \le b - 1$.

If the initial values are all nonnegative integers, we could interpret the Pascal array as representing many gamblers with different amounts of initial capital gambling at the same time. The full generality of complex entries will be needed in the last section of this paper.

LEMMA 1: If

$$\vec{X}_0 = \sum_{k=1}^{b-1} \alpha_k \vec{e}_k,$$

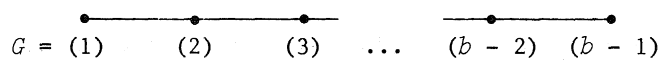
then

$$F(n, b, \vec{X}_0, i) = \sum_{k=1}^{b-1} \alpha_k F(n, b, \vec{e}_k, i).$$

PROOF: This is true for $n = 0$ since the 0th column of P.A. $(b, \vec{X}_0)$ consists of the coordinates $(\alpha_1, \ldots, \alpha_{b-1})$ of $\vec{X}_0$. The recursive definition (1) can then be used to establish the result for all n. □

2. GRAPH THEORY AND RECURRENCE RELATIONS

To learn more about Pascal arrays, it is useful to consider the labeled graph



A gambler could keep track of his gains and losses by moving a marker in a "random walk" along the vertices of this graph. (He would have to leave the graph when he achieved victory or ruin.)

The associated *adjacency matrix* A_b is the $(b - 1) \times (b - 1)$ matrix with

$$A_b(j, i) = \begin{cases} 1 \text{ if vertices } (j) \text{ and } (i) \text{ are connected by an edge,} \\ 0 \text{ otherwise.} \end{cases}$$

For example,

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

LEMMA 2: [2, Lemma 2.5, p. 11] For $n \ge 1$, the (j, i)-entry of the matrix power A_b^n equals the number of paths of G of length n starting at vertex (j) and ending at vertex (i). □

In Pascal array terminology, $A_b(j, i) = F(n, b, \vec{e}_j, i)$.

The *characteristic polynomial* of A_b is

$$P_b(\lambda) = \det(A_b - \lambda I_b),$$

where I_b is the $(b - 1) \times (b - 1)$ identity matrix. We have $P_2 = -\lambda$, $P_3 = \lambda^2 - 1$, and, in general, we expand the determinant by its first row to obtain the important recursive formula

$$P_k(\lambda) = -\lambda P_{k-1}(\lambda) - P_{k-2}(\lambda). \tag{2}$$

Note the similarity of this definition to that of the Fibonacci polynomials. Consistent with (2), we define

$$P_1(\lambda) = -P_3(\lambda) - \lambda P_2(\lambda) = 1 \quad \text{and} \quad P_0(\lambda) = -P_2(\lambda) - \lambda P_1(\lambda) = 0.$$

In Diagram 2, we give a chart of the $P_k(\lambda)$ for $0 \le k \le 10$.

k	$P_k(\lambda)$
0	0
1	1
2	$-\lambda$
3	$\lambda^2 - 1$
4	$-\lambda^3 + 2\lambda$
5	$\lambda^4 - 3\lambda^2 - 1$
6	$-\lambda^5 + 4\lambda^3 - 3\lambda$
7	$\lambda^6 - 5\lambda^4 + 6\lambda^2 - 1$
8	$-\lambda^7 + 6\lambda^5 - 10\lambda^3 + 4\lambda$
9	$\lambda^8 - 7\lambda^6 + 15\lambda^4 - 10\lambda^2 + 1$
10	$-\lambda^9 + 8\lambda^7 - 21\lambda^5 + 20\lambda^3 - 5\lambda$

Diagram 2

LEMMA 3: $P_k(\lambda) = i^{1-k}F_k(-i\lambda)$, where the F_k are the Fibonacci polynomials.

PROOF: By induction.

$$P_0(\lambda) = 0 = i^1F_0(-i\lambda) \quad \text{and} \quad P_1(\lambda) = 1 = i^0F_1(-i\lambda).$$

For $k \ge 2$,

$$\begin{aligned} P_k(\lambda) &= -\lambda P_{k-1}(\lambda) - P_{k-2}(\lambda) \quad \text{(by inductive assumption)} \\ &= (-\lambda)i^{1-(k-1)}F_{k-1}(-i\lambda) - i^{1-(k-2)}F_{k-2}(-i\lambda) \\ &= i^{1-k}[(-i\lambda)F_{k-1}(-i\lambda) - i^2F_{k-2}(-i\lambda)] \\ &= i^{1-k}[(-i\lambda)F_{k-1}(-i\lambda) + F_{k-2}(-i\lambda)] \\ &= i^{1-k}F_k(-i\lambda). \quad \square \end{aligned}$$

LEMMA 4: $P_k(\lambda)$ is a polynomial with integer coefficients having degree $k - 1$ and leading coefficient $(-1)^{k-1}$.

PROOF: These statements follow from Eq. (2) by induction. □

The Cayley-Hamilton Theorem [4, Cor. 2, p. 244] states that $P_b(A_b)$ equals the zero matrix. Then for any $m \ge 0$, $A_b^m \cdot P_b(A_b)$ equals the zero matrix. Let

$$P_b(\lambda) = \sum_{k=0}^{b-1} \beta_k \lambda^k.$$

Thus

$$\sum_{k=0}^{b-1} \beta_k A_b^{m+k} \text{ equals the zero matrix for all } m \ge 0.$$

Looking at individual entries,

$$\sum_{k=0}^{b-1} \beta_k A_b^{m+k}(j, i) = 0 \text{ for all } m \ge 0,\ 0 < i,\ j < b.$$

By Lemma 2, this is equivalent to

$$\sum_{k=0}^{b-1} \beta_k F(m + k, b, \vec{e}_j, i) = 0 \text{ for all } m \ge 0,\ 0 < i,\ j < b. \tag{3}$$

When a sequence satisfies a linear recurrence relation such as (3), we say that

$$P_b(\lambda) = \sum_{k=0}^{b-1} \beta_k \lambda^k = 0$$

is its *auxiliary equation.*

We have proved the following.

THEOREM 5: Every row $\{F(n, b, \vec{e}_j, i)\}_{n=0}^{\infty}$ of the Pascal array P.A. $(b, \vec{e}_j)$ is a sequence which satisfies a linear recurrence relation with constant coefficients and auxiliary equation $P_b(\lambda) = 0$. □

COROLLARY 6: Every row of *any* Pascal array P.A. $(b, \vec{X}_0)$ satisfies the linear recurrence relation with auxiliary equation $P_b(\lambda) = 0$.

PROOF: This follows from Theorem 5, from Lemma 1, and from the superposition principle for solutions to linear recurrence relations. □

As a consequence of Corollary 6 and Lemma 4, if we know a row of a Pascal array P.A. $(b, \vec{X}_0)$ as far as the $(b-2)$nd column, we can reconstruct the whole row uniquely.

We have not yet derived a closed-form expression for $P_k(\lambda)$. Following [4, pp. 267-70], we write

$$P_{k+1}(\lambda) = -\lambda P_k(\lambda) - P_{k-1}(\lambda)$$
$$P_k(\lambda) = P_k(\lambda)$$

In matrix terms,

$$\begin{bmatrix} P_{k+1}(\lambda) \\ P_k(\lambda) \end{bmatrix} = \begin{bmatrix} -\lambda & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_k(\lambda) \\ P_{k-1}(\lambda) \end{bmatrix}.$$

A long calculation then produces the closed-form expression

$$P_k(\lambda) = \sum_{j=1}^{\left[\frac{k+1}{2}\right]} (-1)^{k-j} \binom{k-j}{j-1} \lambda^{k-(2j-1)}.$$

This parallels [1, Eq. (1.3), p. 409]. And, as in [1, Sec. 3], the matrix

$$M = \begin{bmatrix} -\lambda & -1 \\ 1 & 0 \end{bmatrix}$$

can be made to yield a great number of identities based on the iterative property

$$M^k = \begin{bmatrix} P_{k+1}(\lambda) & -P_k(\lambda) \\ P_k(\lambda) & -P_{k-1}(\lambda) \end{bmatrix}.$$

Unlike the Fibonacci polynomials [5, Theorem 1], the $P_k(\lambda)$ are reducible for $k \geq 3$. Their factors are interesting, and should be a subject of further study.

3. DIVISIBILITY PROPERTIES

In this section we will show how divisibility properties of $\{P_b(\lambda)\}$ similar to those of the Fibonacci polynomials [1, p. 415] follow from the consideration of Pascal arrays. Some of our theorems could also be derived using the above matrix M, but we wish to give proofs in the spirit of the gambler's ruin problem.

LEMMA 7: Let λ_0 be a root of $P_b(\lambda) = 0$. Then the sequence $\{1, \lambda_0, \lambda_0^2, \ldots$ satisfies the linear recurrence relation with auxiliary equation $P_b(\lambda) = 0$.

PROOF: Let

$$P_b(\lambda) = \sum_{k=0}^{b-1} \beta_k \lambda^k.$$

Then for any $m \geq 0$,

$$\sum_{k=0}^{b-1} \beta_k \lambda_0^{k+m} = \lambda_0^m \sum_{k=0}^{b-1} \beta_k \lambda_0^k = 0, \text{ since } \lambda_0 \text{ is a root of } P_b(\lambda) = 0. \ \square$$

LEMMA 8: There exists a vector $\vec{X}_1$ such that P.A. $(b, \vec{X}_1)$ has bottom row $\{1, \lambda_0, \lambda_0^2, \ldots\}$, where λ_0 is a root of $P_b(\lambda) = 0$.

PROOF: Suppose we are given

$$F(0, b, \vec{X}_1, 1) = 1,\ F(1, b, \vec{X}_1, 1) = \lambda_0,\ \ldots,\ F(b-2, b, \vec{X}_1, 1) = \lambda_0^{b-2},$$

and we wish to determine $\vec{X}_1$. Using Eq. (1) (third clause), we can determine $F(0, b, \vec{X}_1, 2)$ through $F(b-3, b, \vec{X}_1, 2)$. Then using Eq. (1) (second clause), we can determine $F(0, b, \vec{X}_1, 3), \ldots, F(b-4, b, \vec{X}_1, 3), \ldots, F(0, b, \vec{X}_1, b-2)$, $F(1, b, \vec{X}_1, b-2)$, and $F(0, b, \vec{X}_1, b-1)$. Thus, $\vec{X}_1$ is determined uniquely. Diagram 3 illustrates this procedure in case $b = 5$.

5					
4	$\lambda_0^3 - 2\lambda_0$	...			
3	$\lambda_0^2 - 1$	$\lambda_0^3 - \lambda_0$	...		
2	λ_0	λ_0^2	λ_0^3	...	
1	1	λ_0	λ_0^2	λ_0^3	...
	0	1	2	3	

Diagram 3

Now we can fill in all of P.A. $(b, \vec{X}_1)$. By Corollary 6, its bottom row satisfies the linear recurrence relation with auxiliary equation $P_b(\lambda) = 0$. By the remark following Corollary 6 and Lemma 7, that row must be $\{1, \lambda_0, \lambda_0^2, \lambda_0^3, \ldots\}$. □

Next we show that $\{1, \lambda_0, \lambda_0^2, \ldots\}$, and indeed all of P.A. $(\vec{X}_1, b)$, can be embedded in a Pascal array of height bc for any integer $c > 0$. It then follows easily that λ_0 is also a root of $P_{bc}(\lambda) = 0$.

If $\vec{X} = \alpha_1\vec{e}_1 + \cdots + \alpha_k\vec{e}_k$, then the *palindrome* $\vec{X}^p$ is defined to be

$$\alpha_k\vec{e}_1 + \cdots + \alpha_1\vec{e}_k.$$

We construct an arbitrary array $G(n, i)$, $n \geq 0$ and $0 < i < bc$, as follows:

$$G(n, i) = \begin{cases} F(n, b, \vec{X}_1, i - 2db) \text{ if } 2db < i < (2d+1)b \text{ and } 0 \leq d \leq \left[\frac{c-1}{2}\right], \\ 0 \text{ if } i \text{ is a multiple of } b, \\ F(n, b, -\vec{X}_1^p, i - (2d-1)b) \text{ if } (2d-1)b < i < 2db \text{ and } 1 \leq d \leq \left[\frac{c}{2}\right]. \end{cases}$$

In Diagram 4, we illustrate this construction in the case $b = 5$, $c = 2$, $\vec{X}_1 = \vec{e}_3$.

$i = bc$								
9	0	0	-1	0	-3	0	-8	0
8	0	-1	0	-3	0	-8	0	-21
7	-1	0	-2	0	-5	0	-13	0
6	0	-1	0	-2	0	-5	0	-13
5	0	0	0	0	0	0	0	0
4	0	1	0	2	0	5	0	13
3	1	0	2	0	5	0	13	0
2	0	1	0	3	0	8	0	21
1	0	0	1	0	3	0	8	0
	0	1	2	3	4	5	6	7

Diagram 4

LEMMA 9: $G(n, i)$ is a Pascal array of height bc with initial vector

$$\vec{X}_2 = (\vec{X}_1, 0, -\vec{X}_1^p, 0, \ldots).$$

PROOF: This follows by checking definition (1) in the five cases of an entry in an all-zero row, an entry next to an all-zero row, an entry in the interior of one of the copies of P.A. $(\vec{X}_1, b)$ or P.A. $(-\vec{X}_1^p, b)$, and an entry in the top or bottom row of the whole array. □

THEOREM 10: Any root of $P_b(\lambda) = 0$ is a root of $P_{bc}(\lambda) = 0$.

PROOF: We have just seen that if λ_0 is any root of $P_b(\lambda) = 0$, then $\{1, \lambda_0, \lambda_0^2, \ldots\}$ is the bottom row of a Pascal array of height bc. By Corollary 6, the sequence satisfies the linear recurrence relation with auxiliary equation $P_{bc}(\lambda) = 0$. Applying this fact to the subsequence $\{1, \lambda_0, \ldots, \lambda_0^{bc-1}\}$, we have that

$$P_{bc}(\lambda_0) = 0. \ \square$$

THEOREM 11: $P_b(\lambda)$ divides $P_{bc}(\lambda)$, with quotient a polynomial $Q(\lambda)$ with integer coefficients and leading coefficient ±1.

PROOF: By Theorem 10, $P_b(\lambda)$ divides $P_{bc}(\lambda)$. Let the quotient be $Q(\lambda)$. Define

$$Q(\lambda) = \sum_{k=0} \alpha_k \lambda^k, \quad P_b(\lambda) = \sum_{k=0}^{b-2} \beta_k \lambda^k \pm \lambda^{b-1}, \text{ and } P_{bc}(\lambda) = \sum_{k=0} \gamma_k \lambda^k \pm \lambda^{bc-1}.$$

The form of these last two expressions is dictated by Lemma 4. By multiplication of leading coefficients, $(\pm 1)\alpha_{bc-b} = \pm 1$, which implies that $\alpha_{bc-b} = \pm 1$. Suppose we have proved that $\alpha_{bc-b}, \alpha_{bc-b-1}, \ldots, \alpha_{bc-b-k}$ are integers. Then by polynomial multiplication,

$$\alpha_{bc-b-(k+1)}(\pm 1) + \alpha_{bc-b-k}\beta_{bc-2} + \cdots + = \gamma_{bc-(k+2)}.$$

Thus $\alpha_{bc-b-(k+1)}$ must also be an integer. This completes the proof by induction. □

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