From Eq. (26), we obtain the following equation determining the value of $F_c(a, x)$:

(27) $$F_c - x = \frac{a}{F_c},$$

whence:

(28) $$F_c^2 - xF_c - a = 0.$$

This equation is identical to the one which determines the continued square root $R(2, a, x)$, and correspondingly

(29) $$F_c(a, x) = R(2, a, x).$$

An interesting result of Eq. (28) is that in the limit that $x \to 0$, we find

(30) $$\lim_{x \to 0} F_c(a, x) = a^{\frac{1}{2}},$$

which does not seem obvious from the definition of $F_c(a, x)$ by Eq. (26).

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GENERALIZED FERMAT AND MERSENNE NUMBERS

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1. INTRODUCTION

The numbers $F_n = 1 + 2^{2^n}$ and $M_p = 2^p - 1$, where n is a nonnegative integer and p is a prime, are called Fermat and Mersenne numbers, respectively. Properties of these numbers have been studied for centuries and most of them are well known. At present, the number of known Fermat and Mersenne primes are five and twenty-seven, respectively. It is well known that if $2^n - 1 = p$, a prime, then n is a prime. It is quite easy to show that if $2^n - 1 = pq$, p and q are primes, then either n is a prime or $n = v^2$, where v is a prime. Thus

$$2^{v^2} - 1 = pq = (2^v - 1)(2^{v(v-1)} + \cdots + 2^v + 1),$$

where $2^v - 1 = p$ is a Mersenne prime. This leads to the following definition. Let k and n be positive integers. The number $L(k, n)$ is defined as follows:

$$L(k, n) = 1 + 2^n + (2^n)^2 + \cdots + (2^n)^{k-1}.$$

The purpose of this paper is to study the numbers $L(k, n)$, which contain both the Fermat, $L(2, 2^n)$, and Mersenne, $L(k, 1)$, numbers. We will show that while $L(k, n)$ possesses many interesting properties, there remain unanswered some very elementary questions about this class of numbers.

2. PRIME NUMBERS OF THE FORM $L(k, n)$

In this section, we shall show that if $L(k, n)$ is a prime, then either $L(k, n)$ is a Mersenne prime, or $n = p^i$ and $k = p$. But first we need a lemma which follows from Theorem 10 in [4, p. 17].

LEMMA 1: If $(a, b) = d$, then $(2^a - 1, 2^b - 1) = 2^d - 1$.

THEOREM 1: If $L(k, n)$ is a prime, then either $L(k, n)$ is a Mersenne prime, or $n = p^i$ and $k = p$, where p is a prime and i is a positive integer.

PROOF: If $n = 1$, then $L(k, 1)$ is a Mersenne prime. So suppose $n > 1$. If k is not a prime, then $k = ab$, where $a > 1$ and $b > 1$. Then

$$\begin{aligned} L(k, n)(2^n - 1) &= 2^{nk} - 1 = 2^{n(ab)} - 1 = (2^{na})^b - 1 \\ &= (2^{na} - 1)(2^{na(b-1)} + \cdots + 2^{na} + 1) \\ &= (2^n - 1)(2^{n(a-1)} + \cdots + 2^a + 1) \\ &\quad \cdot (2^{na(b-1)} + \cdots + 2^{na} + 1). \end{aligned}$$

Thus, cancelling $(2^n - 1)$ from both sides, $L(k, n)$ is not a prime; a contradiction. Thus $k = p$ for some prime p.

Next we wish to show that $n = p^i$. Suppose $n = p_1^{d_1} \ldots p_j^{d_j}$ and $p \neq p_k$ for any k. Then

$$L(k, n)(2^n - 1) = (2^n)^p - 1 = (2^p - 1)(2^{p(n-1)} + \ldots + 1).$$

Since $(p, n) = 1$, by Lemma 1, $(2^p - 1, 2^n - 1) = 1$. It follows that

$$(2^p - 1) | L(k, n).$$

If $(2^p - 1)$ is a proper divisor of $L(k, n)$, then $L(k, n)$ is not a prime; a contradiction. If $2^p - 1 = L(k, n)$, then

$$1 + 2 + \cdots + 2^{p-1} = 1 + 2^n + \cdots + (2^n)^{p-1};$$

impossible, since $n > 1$. Thus $p = p_i$ for some i.

Finally, suppose n has more than one prime factor, say $n = p^a x$, $x > 1$. Hence

$$\begin{aligned} L(k, n)(2^n - 1) &= (2^n)^p - 1 = (2^{p^{a+1}})^x - 1 \\ &= (2^{p^{a+1}} - 1)(\cdots) = (2^{p^a} - 1)(\cdots)(\cdots). \end{aligned}$$

Since $(n, p^{a+1}) = p^a$, it follows from Lemma 1 that

$$(2^n - 1, 2^{p^{a+1}} - 1) = 2^{p^a} - 1.$$

Thus

$$(2^{p^a(p-1)} + \cdots + 1) | L(k, n).$$

If $(2^{p^a(p-1)} + \cdots + 1)$ is a proper divisor of $L(k, n)$, then $L(K, n)$ is not a prime; a contradiction. On the other hand,

$$(2^{p^a(p-1)} + \cdots + 1) \neq L(k, n) = 1 + 2 \;\; + \cdots + (2^n)^{p-1}$$

because $n > p^a$. Thus $n = p^a$ and this completes the proof.

For the remainder of this paper, we shall employ the following notation:

$$L(p^i) = 1 + 2^{p^i} + (2^{p^i})^2 + \cdots + (2^{p^i})^{p-1}.$$

REMARK: By looking at the known factors of $2^n - 1$, the following are prime numbers: $L(3), L(3^2)$, and $L(7)$. The numbers $L(3^3), L(5), L(5^2), L(7^2), L(11), L(13)$, and $L(19)$ are *not* primes.

Motivated by the properties of Fermat and Mersenne numbers, we shall investigate the numbers $L(p^i)$ and present a list of unanswered problems concerning $L(p^i)$.

Problem 1. Determine for which primes p there exists a positive integer i for which $L(p^i)$ is a prime.

Problem 2. For each prime p, determine all the primes of the form $L(p^i)$.

3. RELATIVELY PRIME

It is well known [4, pp. 13 and 18] that each pair of Fermat numbers (also the Mersenne numbers) are relatively prime. We show below that for $i \neq j$, $L(p^i)$ and $L(p^j)$ are relatively prime for any prime p.

The proof of the following lemma can be deduced from Theorem 48 [4, p. 105].

LEMMA 2: For each prime p and each positive integer i, $p \nmid L(p^i)$.

THEOREM 2: The numbers $L(p^i)$ and $L(p^{i+k})$ are relatively prime, if $k > 0$.

PROOF: First we show that for any positive integer j, the numbers $L(p^j)$ and $(2^{p^j} - 1)$ are relatively prime. Suppose $m = (L(p^j), (2^{p^j} - 1))$. Since

$$m \mid 2^{p^j} - 1,\ m \mid 2^{p^j n} - 1$$

for any positive integer n. Thus

$$m \mid (2^{p^j(p-1)} - 1) + (2^{p^j(p-2)} - 1) + \cdots + (2^{p^j} - 1) + (1 - 1)$$

implies that $m \mid L(p^j) - p$. Hence $m \mid p$ implies $m = 1$ or p. By Lemma 2, $p \nmid L(p^j)$ and thus $m = 1$. Now

$$\begin{aligned} 2^{p^{i+k}} - 1 &= 2^{(p^{i+k-1})p} - 1 = (2^{p^{i+k-1}} - 1)(2^{p^{i+k-1}(p-1)} + \cdots + 1) \\ &= (2^{p^{i+k-1}} - 1)L(p^{i+k-1}) = (2^p - 1)L(p)L(p^2) \cdots L(p^{i+k-1}). \end{aligned}$$

Suppose the g.c.d. of $L(p^i)$ and $L(p^{i+k})$ is d. Since $L(p^i) \mid 2^{p^{i+k}} - 1$, it follows that $d \mid (2^{p^{i+k}} - 1)$. But $L(p^{i+k})$ and $(2^{p^{i+k}} - 1)$ are relatively prime, thus $d = 1$. The proof is complete.

4. PSEUDOPRIMES

Recall that a number n is called a pseudoprime if $n \mid 2^n - 2$. It is well known [4, p. 115] that each of the Fermat and Mersenne numbers is a pseudoprime. We now show that $L(p^i)$ is a pseudoprime for each i. But first a lemma is needed. It is a consequence of Theorem 48 [4, p. 105].

LEMMA 3: For each prime p and each positive integer i, each prime factor of $L(p^i)$ is of the form $1 + kp^{i+1}$ for some positive integer k.

THEOREM 3: For each prime p and each positive integer i, $L(p^i)$ is a pseudoprime.

PROOF: By Lemma 3, each prime factor of $L(p^i)$ is of the form $1 + kp^{i+1}$ for some positive integer k. Thus, there exists a positive integer x such that

$$L(p^i) = 1 + xp^{i+1}$$

and hence

$$L(p^i) - 1 = xp^{i+1}.$$

Now

$$2^{L(p^i)-1} - 1 = 2^{xp^{i+1}} - 1 = (2^{p^{i+1}} - 1)(2^{p^{i+1}(x-1)} + \cdots + 1).$$

Since $L(p^i)(2^{p^i} - 1) = (2^{p^{i+1}} - 1)$, it follows that

$$L(p^i) \mid (2^{p^{i+1}} - 1)$$

and hence, from above,

$$L(p^i) \mid 2^{L(p^i)-1} - 1.$$

Thus $L(p^i)$ is a pseudoprime.

5. POWERS OF $L(p^i)$

The Fibonacci sequence is defined recursively:

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

It was shown in [1] and [6] that the only Fibonacci squares are F_1, F_2, and F_{12}. In [3] necessary conditions are given for Fibonacci numbers that are prime powers of an integer. It is well known that each Fermat or Mersenne number [2] cannot be written as a power (greater than one) of an integer. We shall show that $L(3^i)$ also shares this property. However, whether $L(p^i)$, for arbitrary p, has this property or not is an open question.

THEOREM 4: Let q and j be positive integers. If $L(3^i) = q^j$, then $j = 1$.

PROOF: In fact, we prove a bit more. Let n be a positive integer. Suppose $1 + 2^n + (2^n)^2 = q^j$ and $j > 1$. Note that q is odd.

Case 1. $j > 1$ is odd. Let $x = 2^n$. Then

$$x(1 + x) = q^j - 1 = (q - 1)(q^{j-1} + \cdots + q + 1).$$

Since $(q - 1)$ is even, $L = (q^{j-1} + \cdots + 1)$ is odd. It follows that

$$x \mid (q - 1) \quad \text{and} \quad x \leq q - 1.$$

Hence $x + 1 \leq q < L$; a contradiction.

Case 2. $j > 1$ is even. It suffices to take $j = 2$. Thus

$$1 + 2^n + (2^n)^2 = q^2 \quad \text{or} \quad 2^n(1 + 2^n) = q^2 - 1 = (q - 1)(q + 1).$$

Since both $(q - 1)$ and $(q + 1)$ are even, and $1 + 2^n$ is odd, it follows that

$$q - 1 = 2^a Q \quad \text{and} \quad q + 1 = 2^b V,$$

where both Q and V are odd and $a + b = n$. Now

$$2 = (q + 1) - (q - 1) = 2^b V - 2^a Q = 2(2^{b-1}V - 2^{a-1}Q).$$

Hence $1 = 2^{b-1}V - 2^{a-1}Q$. It is clear that either $a = 1$ or $b = 1$. Suppose $a = 1$. Then $1 + Q = 2^{b-1}V$. If $Q = 1$, then $2 = 2^{b-1}V$ implies that $V = 1$, and this cannot happen. Thus $Q > 1$. Now

$$2^n(1 + 2^n) = 2Q2^{n-1}V = 2^n QV,\ 1 + 2^n = QV = V(2^{n-2}V - 1),$$

and

$$V + 1 = 2^{n-2}V^2 - 2^n = 2^{n-2}(V^2 - 2^2) = 2^{n-2}(V - 2)(V + 2).$$

Clearly, this is a contradiction. The case $b = 1$ is similar. This completes the proof.

We can also show that, for $p = 5, 7, 11$, $L(p^i)$ is not the power (greater than one) of any positive integer. The general case has, so far, eluded our investigation. It is so intriguing that we shall state it as a conjecture.

CONJECTURE 1: Let p be an arbitrary prime and q and j be positive integers. If $L(p^i) = q^j$, then $j = 1$.

6. COMMENTS

Even though each Fermat or Mersenne number is not the power (greater than one) of an integer, it is not known whether they are square-free. Naturally, we make a similar conjecture.

CONJECTURE 2: For each prime p and positive integer i, the number $L(p^i)$ is square-free.

REMARK: It has been shown in [5] that the congruence $2^{p-1} \equiv 1 \pmod{p^2}$ is closely related to the square-freeness of the Fermat and Mersenne numbers. We have shown, by a similar method, that this is also the case for the numbers $L(p^i)$.

It is well known that $(p,\ 2^p - 1) = 1$ and $(n,\ 1 + 2^{2^n}) = 1$. Since the prime divisors of $L(p^i)$ are of the form $1 + kp^{i+1}$ [4, p. 106], it follows that

$$(i,\ L(p^i)) = 1.$$

Finally, we see that while $L(p^i)$ possesses many interesting properties, there remain unanswered some very elementary questions about this class of numbers.

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FIBONACCI NUMBERS OF GRAPHS

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1. INTRODUCTION

According to [1, p. 45], the total number of subsets of $\{1, \ldots, n\}$ such that no two elements are adjacent is F_{n+1}, where F_n is the nth Fibonacci number, which is defined by

$$F_0 = F_1 = 1,\ F_n = F_{n-1} + F_{n-2}.$$

The sequence $\{1, \ldots, n\}$ can be regarded as the vertex set of the graph P_n in Figure 1. Thus, it is natural to define the Fibonacci number $f(X)$ of a (simple) graph X with vertex set V and edge set E to be the total number of subsets S of V such that any two vertices of S are not adjacent.

The Fibonacci number of a graph X is the same as the number of complete (induced) subgraphs of the complement graph of X. (Our terminology covers the empty graph also.)