Assume that $(t+1, n-1, f) \in A$. Then β' is odd. Set $\tilde{E}_{t,\beta'+1} = E_{t+1,\beta'} + 1$. Since $E_{t+1,\beta'} \geq n-1$, we have

(3) $$\tilde{E}_{t,\beta'+1} \geq n.$$

Consequently,

(4) $$\tilde{E}_{t,\beta'+1} > E_{t,\alpha}.$$

By the maximality of α, the minimality of β, and (4), we conclude that $E_{t,\beta'+1} > 0$ (and, of course, $E_{t,\beta'+1} = \tilde{E}_{t,\beta'+1}$). But $\beta'+1$ is even, β is odd, and $\beta'+1 \leq \beta$. Hence, we also have $\beta'+1 < \beta$. $\beta'+1 < \beta$ and (3) contradict the minimality of β. We conclude that $(t+1, n-1, f) \notin A$.

We have shown that, in both Case b.1 and Case b.2, statement (b) holds.

c. Suppose $\underline{(t, n, f) \notin A}$. If $n = 0$, statement (c) is vacuous. So assume $n > 0$. Observe that β is even and that $\beta > 0$. Let $\gamma = \gamma(t, n, f)$. If $\gamma = 0$, then $E_{t,\gamma+1} > 0$. If $\gamma > 0$, then γ even and $E_{t+1,\gamma} > 0$ imply that $E_{t,\gamma+1} > 0$ by Lemma 6.2. Thus, in either case, $E_{t,\gamma+1} > 0$. So $\underline{\gamma + 1 \leq \alpha}$ by the maximality of α, the minimality of β, and the fact that $\alpha + 1 < \beta$.

Now $0 < E_{t,\gamma+1} \leq E_{t,\alpha} < n$ and γ even imply that

(5) $$n - f(t, n) > E_{t+1,\gamma}$$

by (a) of Lemma 6.1. Since $n \leq E_{t,\beta} = E_{t+1,\beta-1} + 1$, $n - 1 \leq E_{t+1,\beta-1}$. Combine this with (5) to get

$$\underline{E_{t+1,\gamma} < n - u \leq E_{t+1,\beta-1} \ \forall u \ni 1 \leq u \leq f(t, n).}$$

Thus, $\beta(t+1, n-u, f)$ is odd $\forall u \ni 1 \leq u \leq f(t, n)$. We have shown that

$$\underline{F(t, n, f) \subseteq A,}$$

which verifies statement (c). Q.E.D.

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FIBONACCI-CAYLEY NUMBERS

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Horadam [2] defined and studied in detail the generalized Fibonacci sequence defined by

(1) $$H_n = H_{n-1} + H_{n-2} \qquad (n > 2),$$

with $H_1 = p$, $H_2 = p + q$, p and q being arbitrary integers. In a later article [3] he defined Fibonacci and generalized Fibonacci quaternions as follows, and established a few relations for these quaternions:

(2) $$P_n = H_n + H_{n+1}i_1 + H_{n+2}i_2 + H_{n+3}i_3,$$

and

(3) $$Q_n = F_n + F_{n+1}i_1 + F_{n+2}i_2 + F_{n+3}i_3,$$

where

$$i_1^2 = i_2^2 = i_3^2 = -1,\ i_1 i_2 = -i_2 i_1 = i_3,$$

and

$$i_2 i_3 = -i_3 i_2 = i_1,\ i_3 i_1 = -i_1 i_3 = i_2,$$

and $\{F_n\}$ is the Fibonacci sequence defined by

$$F_1 = F_2 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \qquad (n > 2).$$

He also defined the conjugate quaternion as

(4) $$\overline{P}_n = H_n - H_{n+1}i_1 - H_{n+2}i_2 - H_{n+3}i_3,$$

and $\overline{Q}_n$ in a similar way. M. N. S. Swamy [5] obtained some additional relations for these quaternions.

In Section 1 of this paper we define the Fibonacci and generalized Fibonacci-Cayley numbers. In Section 2, we obtain a further generalization of these numbers as well as of the complex Fibonacci numbers and Fibonacci quaternions discussed in [3] and [5].

SECTION 1

We call

(5) $$R_n = F_n + F_{n+1}i_1 + \cdots + F_{n+7}i_7$$

and

(6) $$S_n = H_n + H_{n+1}i_1 + \cdots + H_{n+7}i_7,$$

where

$$i_1^2 = \cdots = i_7^2 = -1,\ i_1 i_2 = i_3 = -i_2 i_1,$$
$$i_2 i_3 = i_1 = -i_3 i_2,\ i_3 i_1 = i_2 = -i_1 i_3,$$

and six similar sets of six relations with 1, 2, 3 replaced by 1, 4, 5; 6, 2, 4; 6, 5, 3; 7, 2, 5; 7, 3, 4; and 1, 7, 6, respectively (see [1]), nth Fibonacci and generalized Fibonacci-Cayley numbers, respectively. We define conjugate Cayley numbers as

(7) $$\overline{S}_n = H_n - H_{n+1}i_1 - \cdots - H_{n+7}i_7,$$

and $\overline{R}_n$ in a similar way, so that

$$S_n\overline{S}_n = \sum_{i=0}^{7} H_{n+i}^2 = [H_n^2 + H_{n+1}^2 + H_{n+2}^2 + H_{n+3}^2] + [H_{n+4}^2 + H_{n+5}^2 + H_{n+6}^2 + H_{n+7}^2]$$

$$= P_n\overline{P}_n + P_{n+4}\overline{P}_{n+4} = 3[(2p - q)H_{2n+3} - (p^2 - pq - q^2)F_{2n+3}]$$
$$+ 3[(2p - q)H_{2n+11} - (p^2 - pq - q^2)F_{2n+11}]$$

(using Eq. 15 of [5])

(8) $$= 3[(2p - q)(H_{2n+3} + H_{2n+11}) - (p^2 - pq - q^2)(F_{2n+3} + F_{2n+11})].$$

$S_n + \overline{S}_n = 2H_n$ implies

(9) $$S_n^2 = 2H_n S_n - S_n\overline{S}_n.$$

Since

(10) $$H_{m+n+1} = F_{m+1}H_{n+1} + F_m H_n = F_{n+1}H_{m+1} + F_n H_m$$

(see [2]), we have

$$\begin{aligned} F_{m+1}S_{n+1} + F_m S_n &= (F_{m+1}H_{n+1} + F_m H_n) + \cdots + (F_{m+1}H_{n+4} + F_m H_{n+3})i_3 \\ &\quad + (F_{m+1}H_{n+5} + F_m H_{n+4})i_4 + \cdots + (F_{m+1}H_{n+8} + F_m H_{n+7})i_7 \\ &= H_{m+n+1} + H_{m+n+2}i_1 + \cdots + H_{m+n+8}i_7 \\ &= S_{m+n+1}, \end{aligned}$$

so that

(11) $$S_{m+n+1} = F_{m+1}S_{n+1} + F_m S_n = F_{n+1}S_{m+1} + F_n S_m.$$

This implies

$$S_{2n+1} = F_{n+1}S_{n+1} + F_n S_n$$

and

$$S_{2n} = F_{n+1}S_n + F_n S_{n-1} = F_n S_{n+1} + F_{n-1}S_n.$$

Again, since

(12) $$H_{n+1} = qF_n + pF_{n+1}$$

(Eq. 7 of [2]), we have

$$\begin{aligned} H_{m+1}S_{n+1} + H_m S_n &= (qF_m + pF_{m+1})S_{n+1} + (qF_{m-1} + pF_m)S_n \\ &= p(F_{m+1}S_{n+1} + F_m S_n) + q(F_m S_{n-1} + F_{m-1}S_n) \\ &= pS_{m+n+1} + qS_{m+n} \text{ [by (11)]}. \end{aligned}$$

Using (8) and (12) above and Eq. 17 of [5], we get

(14) $$\begin{aligned} S_n\overline{S}_n &= 3[p^2F_{2n+3} + 2pqF_{2n+2} + q^2F_{2n+1} + p^2F_{2n+11} + 2pqF_{2n+10} + q^2F_{2n+9}] \\ &= 3[p^2(F_{2n+3} + F_{2n+11}) + 2pq(F_{2n+2} + F_{2n+10}) + q^2(F_{2n+1} + F_{2n+9})]. \end{aligned}$$

Hence

(15) $$\begin{aligned} S_n\overline{S}_n + S_{n-1}\overline{S}_{n-1} &= 3[p^2(F_{2n+3} + F_{2n+11} + F_{2n+1} + F_{2n+9}) \\ &\quad + 2pq(F_{2n+2} + F_{2n+10} + F_{2n} + F_{2n+8}) \\ &\quad + q^2(F_{2n+1} + F_{2n+9} + F_{2n+1} + F_{2n+7})] \\ &= 3[p^2(L_{2n+2} + L_{2n+10}) + 2pq(L_{2n+1} + L_{2n+9}) \\ &\quad + q^2(L_{2n} + L_{2n+8})], \end{aligned}$$

since $L_n = F_{n-1} + F_{n+1}$, where $\{L_n\}$ is the Lucas sequence defined by

$$L_1 = 1,\ L_2 = 3,\ L_n = L_{n-1} + L_{n-2} \quad (n > 2).$$

From (9), (13), and (15), we have

$$\begin{aligned} S_n^2 + S_{n-1}^2 &= 2(H_n S_n + H_{n-1}S_{n-1}) - (S_n\overline{S}_n + S_{n-1}\overline{S}_{n-1}) \\ &= 2(pS_{2n-1} + qS_{2n-2}) - 3[p^2(L_{2n+2} + L_{2n+10}) + 2pq(L_{2n+1} + L_{2n+9}) \\ &\quad + q^2(L_{2n} + L_{2n+8})]. \end{aligned}$$

Analogous to Eq. 16 of [2], we have

(17) $$\{2S_{n+1}S_{n+2}\}^2 + \{S_n S_{n+3}\}^2 = \{2S_{n+1}S_{n+2} + S_n\}^2.$$

Using (11), we can establish the identity analogous to Eq. 17 of [2]:

(18) $$\frac{S_{n+t} + (-1)\ S_{n-t}}{S_n} = F_{t-1} + F_{t+1}.$$

If $p = 1$, $q = 0$, then we have the Fibonacci sequence $\{F_n\}$ and the corresponding Cayley number R_n for which we may write the following results:

(19) $$R_n\overline{R}_n = \overline{R}_nR_n = 3(F_{2n+3} + F_{2n+11}).$$

(20) $$R_n\overline{R}_n + R_{n-1}\overline{R}_{n-1} = 3(L_{2n+2} + L_{2n+10}).$$

(21) $$R_n^2 + R_{n-1}^2 = 2R_{2n-1} - 3(L_{2n+2} + L_{2n+10}).$$

Similar results may be obtained for the Lucas numbers and the corresponding Cayley numbers by letting $p = 1$ and $q = 2$ in the various results derived above.

SECTION 2

A. The following facts about composition algebras over the field of real numbers (the details of which can be found in [4]) are needed to obtain further generalization of complex Fibonacci numbers, Fibonacci quaternions, and Fibonacci-Cayley numbers.

1. The 2-dimensional algebra over the field R of real numbers with basis $\{1, i_1\}$ and multiplication table

	1	i_1
1	1	i_1
i_1	i_1	$-\alpha$

(α being any nonzero real number).

We denote this algebra by $C(\alpha)$. The conjugate of $x = a_0 + a_1i_1$ is $\overline{x} = a_0 - a_1i_1$ and $x\overline{x} = \overline{x}x = a_0^2 + \alpha a_1^2$.

2. The 4-dimensional algebra (over R) with basis $\{1, i_1, i_2, i_3\}$ and multiplication table

	1	i_1	i_2	i_3
1	1	i_1	i_2	i_3
i_1	i_1	$-\alpha$	i_3	$-\alpha i_2$
i_2	i_2	$-i_3$	$-\beta$	βi_1
i_3	i_3	αi_2	$-\beta i_1$	$-\alpha\beta$

(α, β any nonzero real numbers).

We denote this algebra by $C(\alpha, \beta)$. The conjugate of $x = a_0 + a_1i_1 + a_2i_2 + a_3i_3$ is $\overline{x} = a_0 - a_1i_1 - a_2i_2 - a_3i_3$ and $x\overline{x} = \overline{x}x = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2$.

3. The 8-dimensional algebra (over R) with basis $\{1, i_1, \ldots, i_7\}$ and multiplication table

	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_1	$-\alpha$	i_3	$-\alpha i_2$	i_5	$-\alpha i_4$	$-i_7$	αi_6
i_2	$-i_3$	$-\beta$	βi_1	i_6	i_7	$-\beta i_4$	$-\beta i_5$
i_3	αi_2	$-\beta i_1$	$-\alpha\beta$	i_7	$-\alpha i_6$	βi_5	$-\alpha\beta i_4$
i_4	$-i_5$	$-i_6$	$-i_7$	$-\gamma$	γi_1	γi_2	γi_3
i_5	αi_4	$-i_7$	αi_6	$-\gamma i_1$	$-\alpha\gamma$	$-\gamma i_3$	$\gamma\alpha i_2$
i_6	i_7	βi_4	$-\beta i_5$	$-\gamma i_2$	γi_3	$-\gamma\beta$	$-\gamma\beta i_1$
i_7	$-\alpha i_6$	βi_5	$\alpha\beta i_4$	$-\gamma i_3$	$-\gamma\alpha i_2$	$\gamma\beta i_1$	$-\alpha\beta\gamma$

(α, β, γ any nonzero real numbers).

We denote this algebra by $C(\alpha, \beta, \gamma)$. The conjugate of $x = a_0 + a_1 i_1 + \cdots + a_7 i_7$ is $\bar{x} = a_0 - a_1 i_1 - \cdots - a_7 i_7$ and $x\bar{x} = \bar{x}x = (a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2) + \gamma(a_4^2 + \alpha a_5^2 + \beta a_6^2 + \alpha\beta a_7^2)$.

B. Next we shall consider the following generalizations of H_n, F_n, and L_n, respectively:

$$h_n\colon h_1 = p,\ h_2 = bp + cq,\ h_n = bh_{n-1} + ch_{n-2} \quad (n > 2)$$
$$f_n\colon f_1 = 1,\ f_2 = b,\ f_n = bf_{n-1} + cf_{n-2} \quad (n > 2)$$
$$\ell_n\colon \ell_1 = b,\ \ell_2 = b^2 + 2c,\ \ell_n = b\ell_{n-1} + c\ell_{n-2} \quad (n > 2)$$

(b, c, p, q being integers).

Then we have the following various relations:

$$h_n = pf_n + qcf_{n-1}$$
$$\ell_n = f_{n+1} + cf_{n-1}$$
$$ph_{2n-2} + cqh_{2n-3} = h_{n-1}(ch_{n-2} + h_n)$$
$$ch_n^2 + h_{n+1}^2 = ph_{2n+1} + cqh_{2n} = (2p - bq)h_{2n+1} - ef_{2n+1},$$

where $e = p^2 - bpq - cq^2$.

$$h_n h_{n+1} - c^2 h_{n-2} h_{n-1} = b(ph_{2n-1} + qch_{2n-2})$$
$$h_{n+1}^2 - c^2 h_{n-1}^2 = b(ph_{2n} + cqh_{2n-1}) = b(2p - bq)h_{2n} - bef_{2n}$$
$$h_{n-1}h_{n+1} - h_n^2 = (-c)^n e$$
$$f_{n-1}f_{n+1} - f_n^2 = (-c)^n$$
$$h_{n+t} = ch_{n-1}f_t + h_n f_{t+1} = ch_{t-1}f_n + h_t f_{n+1}$$
$$\frac{h_{n+t} - (-c)^{t+1}h_{n-t}}{h_n} = cf_{t-1} + f_{t+1}.$$

We now define the nth generalized complex Fibonacci number d_n as the element $h_n + h_{n+1}i_1$ of the algebra $C(1/c)$; the nth generalized Fibonacci quaternion p_n as the element $h_n + h_{n+1}i_1 + h_{n+2}i_2 + h_{n+3}i_3$ of the algebra $C(1/c, 1)$; and the nth generalized Fibonacci-Cayley number s_n as the element $h_n + h_{n+1}i_1 + \cdots + h_{n+7}i_7$ of the algebra $C(1/c, 1, 1)$.

The following is a list of relations for these numbers:

$$d_{n-1}d_{n+1} - d_n^2 = (-c)^n e(2 + bi_1).$$
$$d_n\bar{d}_n = \bar{d}_n d_n = h_n^2 + \frac{1}{c}h_{n+1}^2$$
$$= \frac{1}{c}(ph_{2n+1} + cqh_{2n}) = \frac{1}{c}[(2p - bq)h_{2n+1} - ef_{2n+1}]$$
$$= \frac{1}{c}[(p^2 + cq^2)f_{2n+1} + cq(2p - bq)f_{2n}].$$
$$d_n\bar{d}_n + cd_{n-1}\bar{d}_{n-1} = \frac{1}{c}[(p^2 + cq^2)(f_{2n+1} + cf_{2n-1}) + qc(2p - bq)(f_{2n} + cf_{2n-2})]$$
$$= \frac{1}{c}[(p^2 + cq^2)\ell_{2n} + qc(2p - bq)\ell_{2n-1}].$$
$$d_{m+n+1} = f_{m+1}d_{n+1} + cf_m d_n = f_{n+1}d_{m+1} + cf_n d_m.$$
$$h_{m+1}d_{n+1} + ch_m d_n = pd_{m+n+1} + qcd_{m+n}.$$
$$d_n^2 + cd_{n-1}^2 = 2(pd_{2n-1} + qcd_{2n-2}) - \frac{1}{c}[(p^2 + cq^2)\ell_{2n} + qc(2p - bq)\ell_{2n-1}].$$

$$d_{n+1}^2 - c^2 d_{n-1}^2 = -\frac{b^2}{c}(2p - bq)h_{2n+1} + \frac{b^2 e}{c} f_{2n+1} + 2b(ph_{2n+1} + qch_{2n})i_1.$$

$$\frac{d_{n+t} - (-c)^{t+1} d_{n-t}}{d_n} = cf_{t-1} + f_{t+1}.$$

$$p_n \overline{p}_n = d_n \overline{d}_n + d_{n+2}\overline{d}_{n+2} = \frac{1}{c}[(2p - bq)(h_{2n+1} + h_{2n+5}) - e(f_{2n+1} + f_{2n+5})]$$

$$= \frac{1}{c}[(p^2 + cq^2)(f_{2n+1} + f_{2n+5}) + cq(2p - bq)(f_{2n} + f_{2n+4})].$$

$$p_n \overline{p}_n + cp_{n-1}\overline{p}_{n-1} = \frac{1}{c}[(p^2 + cq^2)(\ell_{2n} + \ell_{2n+4}) + cq(2p - bq)(\ell_{2n-1} + \ell_{2n+3})].$$

$$p_{m+n+1} = f_{m+1}p_{n+1} + cf_m p_n = f_{n+1}p_{m+1} + cf_n p_m.$$

$$h_{m+1}p_{n+1} + ch_m p_n = pp_{m+n+1} + qcp_{m+n}.$$

$$p_n^2 + cp_{n-1}^2 = pp_{2n-1} + qcp_{2n-2} - (p_n\overline{p}_n + cp_{n-1}\overline{p}_{n-1}).$$

$$\frac{p_{n+t} - (-c)^{t+1} p_{n-t}}{p_n} = cf_{t-1} + f_{t+1}.$$

$$s_n \overline{s}_n = p_n \overline{p}_n + p_{n+4}\overline{p}_{n+4}.$$

$$s_n \overline{s}_n + cs_{n-1}\overline{s}_{n-1} = p_n\overline{p}_n + cp_{n-1}\overline{p}_{n-1} + p_{n+4}\overline{p}_{n+4} + cp_{n+3}\overline{p}_{n+3}.$$

$$s_{m+n+1} = f_{m+1}s_{n+1} + cf_m s_n = f_{n+1}s_{m+1} + cf_n s_m.$$

$$h_{m+1}s_{n+1} + ch_m s_n = ps_{m+n+1} + qcs_{m+n}.$$

$$s_n^2 + cs_{n-1}^2 = ps_{2n-1} + qcs_{2n-2} - (s_n\overline{s}_n + cs_{n-1}\overline{s}_{n-1}).$$

$$\frac{s_{n+t} - (-c)^{t+1} s_{n-t}}{s_n} = cf_{t-1} + f_{t+1}.$$

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