# GENERALIZATIONS OF SOME PROBLEMS ON FIBONACCI NUMBERS

P. V. SATYANARAYANA MURTHY
*S.K.B.R. College, Amalapuram-533 201, India*
*(Submitted August 1980)*

## 1. INTRODUCTION

In this paper, we obtain generalizations of some problems which have appeared in recent years in *The Fibonacci Quarterly*.

Throughout $\{F_n\}$ denotes the Fibonacci sequence defined by

$$F_1 = F_2 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad (n > 2)$$

and $\{L_n\}$ denotes the Lucas sequence defined by

$$L_1 = 1,\ L_2 = 3, \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \quad (n > 2).$$

Sequences $\{h_n\}$, $\{f_n\}$, and $\{\ell_n\}$ are defined as follows, respectively:

$$h_1 = p,\ h_2 = bp + cq,\ h_n = bh_{n-1} + ch_{n-2} \quad (n > 2)$$

$$f_1 = 1,\ f_2 = b,\ f_n = bf_{n-1} + cf_{n-2} \quad (n > 2)$$

$$\ell_1 = b,\ \ell_2 = b^2 + 2c,\ \ell_n = b\ell_{n-1} + c\ell_{n-2} \quad (n > 2)$$

$$(b,\ c,\ p,\ q \text{ being integers}).$$

Note that for $b = c = p = 1$, $q = 0$ we will have $h_n = f_n$ and for $b = c = 1$ we will have $f_n = F_n$ and $\ell_n = L_n$.

The following relations will be used throughout:

$$h_n = \frac{\ell r^n - ms^n}{r - s}, \qquad f_n = \frac{r^n - s^n}{r - s},$$

$$\ell_n = r^n + s^n, \qquad \ell_n = cf_{n-1} + f_{n+1},$$

$$f_{2n} = f_n \ell_n, \qquad f_{2n} = -c^{2n} f_{-2n},$$

where

$$r + s = b,\ rs = -c,\ \ell = p - sq, \text{ and } m = p - rq.$$

## 2. GENERALIZATIONS

No proofs of the following generalizations are given, since they follow those of the original statements very closely. The original statements are referenced in parentheses, giving the Problem number, Volume number, and Year in which they appeared in *The Fibonacci Quarterly*.

*H-263 (15, 1977):* $\ell_{2mn}^2 \equiv 4c^{2mn} \pmod{\ell_m^2}$.

*H-279 (17, 1979):*

(a) $f_{n+6r}^4 - c^{2r}(\ell_{4r} + c^{2r})(f_{n+4r}^4 - c^{4r} f_{n+2r}^4) - c^{12r} f_n^4 = f_{2r} f_{4r} f_{6r} f_{4n+12r}$.

(b) $f_{n+6r+3}^4 + c^{2r+1}(\ell_{4r+2} - c^{2r+1})(f_{n+4r+2}^4 - c^{4r+2} f_{n+2r+1}^4) - c^{12r+6} f_n^4$
$= f_{2r+1} f_{4r+2} f_{6r+3} f_{4n+12r+6}$.

*LEMMA 1:* $\ell_{3m} - (-c)^m \ell_m = (b^2 + 4c) f_m f_{2m}$.

*LEMMA 2:* $(b^2 + 4c)(f_u^4 - c^{2u-2v} f_v^4) = f_{u-v} f_{u+v} [\ell_{u-v} \ell_{u+v} - 4(-c)^u]$.

*LEMMA 3:* $(-c)^m \ell_{2m} + c^{2m} = (-c)^m f_{3m} / f_m$.

B-271(b) (12, 1974): If $k$ is even, then $\ell_k - 2c^k$ divides

$$h_{(n+2)k} - 2h_{(n+1)k}c^k + h_{nk}c^k.$$

(This generalization was suggested by the referee.)

B-275 (13, 1975): $h_{mn} = \ell_m h_{m(n-1)} - (-c)^m h_{m(n-2)}$.

B-277 (13, 1975): $\ell_{2n(2k+1)} \equiv c^{2nk}\ell_{2n} \pmod{f_{2n}}$.

B-282 (13, 1975): If $c = d^2$ $(d > 0)$, then $2d\ell_n\ell_{n+1}$, $|\ell_{n+1}^2 - c\ell_n^2|$, and $c\ell_{2n} + \ell_{2n+2}$ are the lengths of a right-angled triangle.

B-294 (13, 1975): $h_n\ell_k + h_k\ell_n = 2h_{n+k} + q(-c)^k\ell_{n-k}$.

B-298 (14, 1976): $(b^2 + 4c)h_{2n+3}h_{2n-3} = p^2\ell_{4n} + 2cpq\ell_{4n-1} + q^2c^2\ell_{4n-2} + ec^{2n-3}\ell_6$, where $e = p^2 - bpq - cq^2 = \ell m$.

B-323 (15, 1977): $h_{n+t}^2 - (-c)^t h_n^2 = f_t(ph_{2n+t} + cqh_{2n+t-1})$.

B-342 (15, 1977): $2c^3\ell_{n-1}^3 + b^3\ell_n^3 + 6c\ell_{n+1}^2\ell_{n-1} = (\ell_{n+1} + c\ell_{n-1})^3$.

B-343 (15, 1977): $\sum_{k=1}^{n}[cf_{2k-1}f_{2(n-k)+1} - f_{2k}f_{2(n-k+1)}] = \frac{1}{b^2 + 4c}\left(\frac{4c^2}{b}f_{2n} - bn\ell_{2n+1}\right)$.

B-354 (16, 1978): $h_{n+k}^3 - \ell_k^3 h_n^3 + (-c)^k h_{n-k}[c^{2k}h_{n-k}^2 + 3h_{n+k}h_n\ell_k] = 0$.

B-355 (16, 1978): $h_{n+k}^3 - \ell_{3k}h_n^3 + (-c)^{3k}h_{n-k}^3 = 3e(-c)^n h_n f_k f_{2k}$.

B-379 (17, 1979): $f_{2n} \equiv nb(-c)^{n-1}$ [mod $(b^2 + 4c)$] for $n = 1, 2, \ldots$ .

*****

# A VARIANT OF THE FIBONACCI POLYNOMIALS WHICH ARISES IN THE GAMBLER'S RUIN PROBLEM

MARK E. KIDWELL and CLIFFORD M. HURVICH
*Amherst College, Amherst, MA 01002*
*(Submitted September 1980)*

In examining the gambler's ruin problem (a simple case of random walk) with a finite number of possible states, we were led to consider a sequence of linear recurrence relations that describe the number of ways to reach a given state. These recurrence relations have a sequence of polynomials as their auxiliary equations. These polynomials were unknown to us, but proved exceptionally rich in identities. We gradually noticed that these identities were analogous to well-known identities satisfied by the Fibonacci numbers. A check of back issues of *The Fibonacci Quarterly* then revealed that our sequence of polynomials differed only in sign from the Fibonacci polynomials studied in [1], [5], and several other papers.

In this paper we show, using graph theory and linear algebra, how the gambler's ruin problem gives rise to our sequence of polynomials. We then compare our polynomials to the Fibonacci polynomials and explain why the two sequences satisfy analogous identities. Finally, we use the Pascal arrays introduced in our analysis of gambler's ruin to give a novel proof of the divisibility properties of our sequence.