THE LENGTH OF THE FOUR-NUMBER GAME

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INTRODUCTION

Let D be the operator defined on 4-tuples of nonnegative integers by

$$D(w, x, y, z) = (|w - z|, |w - x|, |x - y|, |y - z|).$$

Given any initial 4-tuple $S = S_0 = (w_0, x_0, y_0, z_0)$, we obtain a sequence $\{S_n\}$, where $S_{n+1} = DS_n$. This sequence is sometimes called the four-number game. The following curious fact seems to have been discovered and rediscovered several times—[3], [4], [5]—$S_n = (0, 0, 0, 0)$ for all sufficiently large n. We can thus make the following definition.

DEFINITION: The length of the sequence $\{S_n\}$, denoted $L(S)$, is the smallest n such that $S_n = (0, 0, 0, 0)$.

A natural question to ask is: "How long can a game continue before all zeros are reached?" Again, it is well known that the length can be arbitrarily long if the numbers in S_n are sufficiently large [4]. One of the easiest ways to see this makes use of the so-called Tribonacci numbers:

$$t_0 = 0,\ t_1 = 1,\ t_2 = 1 \quad \text{and} \quad t_n = t_{n-1} + t_{n-2} + t_{n-3} \quad \text{for } n \geq 3.$$

If we let $T_n = (t_n, t_{n-1}, t_{n-2}, t_{n-3})$, then a simple calculation shows that

$$D^3T_n = 2T_{n-2},$$

and so

$$L(T_n) = 3\left[\frac{n}{2}\right].$$

It has also been noticed that the sequence beginning with some T_n seems to have the longest length of any sequence whose original elements do not exceed t_n. We will prove that this is almost true.

It should be pointed out that if we allow the elements of S_0 to be real, then we can obtain a game of infinite length by taking $S_0 = (r^3, r^2, r, 1)$, where $r = 1.839\ldots$ is the real root of the equation $x^3 - x^2 - x - 1 = 0$ (see [2], [6], [7]). Moreover, this is essentially the only way to obtain a game of infinite length [7]. To obtain a long game with integer entries, we should pick the initial terms to have ratios approximating r [1]. The Tribonacci numbers do this very nicely.

MAIN RESULT

Before proving our main theorem, we need a few easy observations. If

$$|S| = \max(w, x, y, z),$$

then

$$|S_0| \geq |S_1| \geq |S_2| \ \ldots\ .$$

The games having initial elements

$$(w, x, y, z),\ (x, y, z, w),\ (y, z, w, x),\ (z, w, x, y);$$

$$(z, y, x, w);\ (w + k, x + k, y + k, z + k);$$

and $\quad (kw, kx, ky, kz),\ k > 0;$

all have the same length. We now state our main theorem which will be an immediate consequence of Theorem 2.

THEOREM 1: If $|S| \leq |T_n|$, then $L(S) \leq L(T_n) + 1 = 3\left[\frac{n}{2}\right] + 1$.

One of the first things to notice is that $L(S) \leq 6$, unless the elements of S are monotonically decreasing, $w > x > y > z$. [Remember, cyclic permutations and reversals yield equivalent games, so $(5, 7, 12, 2) \sim (2, 5, 7, 12) \sim (12, 7, 5, 2)$, which is monotonically decreasing.) This can be checked by simply calculating the first six S_n if S_0 is not monotonic [1]. Also, if S_n is monotonic decreasing, then S_{n+1} cannot be monotonic increasing. Therefore, in a long game, all of the S_n at the beginning must be monotonic decreasing.

Let $S_n = (w_n, x_n, y_n, z_n)$. We say that S_n is additive if $w_n = x_n + y_n + z_n$. If S_{n-1} is monotonic (decreasing), then a trivial calculation shows that S_n is additive. Thus, although $S = S_0$ may not be additive, $S_1, S_2, \ldots, S_n$ will be additive as long as $S_0, S_1, \ldots, S_{n-1}$ are monotonic.

LEMMA: If $S_1, S_2, \ldots, S_{10}$ are all monotonic (decreasing), S_1 is additive, and $|S_1| \leq t_n$, then either $|S_4| \leq 2t_{n-2}$ or $|S_7| \leq 4t_{n-4}$ or $|S_{10}| \leq 8t_{n-6}$.

PROOF: Write $S_1 = (a + b + c, a, b, c)$ and assume $a + b + c \leq t_n$,

$$|S_4| > 2t_{n-2},\ |S_7| > 4t_{n-4}, \text{ and } |S_{10}| > 8t_{n-6}.$$

Since we know that $S_1 \ldots S_{10}$ are all monotonic, they can be explicitly calculated, and we find that

$$|S_4| = 2b,\ |S_7| = 4a - 4b - 4c, \text{ and } |S_{10}| = 16c - 8b.$$

$|S_4| > 2t_{n-2}$ implies $2b \geq 2t_{n-2} + 2$ or $3b \geq 3t_{n-2} + 3$; $|S_7| > 4t_{n-4}$ implies $a - b - c \geq t_{n-4} + 1$; $|S_{10}| > 8t_{n-6}$ implies $2c - b \geq t_{n-6} + 1$. Adding these three inequalities, we obtain

$$a + b + c \geq 3t_{n-2} + t_{n-4} + t_{n-6} + 5.$$

But since $a + b + c \leq t_n$, we have

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} \geq 3t_{n-2} + t_{n-4} + t_{n-6} + 5.$$

Using the defining relation of the Tribonacci numbers repeatedly, we get

$$2t_{n-3} \geq 2t_{n-3} + 5,$$

which is an obvious contradiction. This proves the lemma.

THEOREM 2: If S_1 is additive and $|S_1| \leq t_n$, then $L(S_1) \leq L(T_n) = 3\left[\frac{n}{2}\right]$, $n \geq 2$.

PROOF: Since S_1 is additive, we may write $S_1 = (a + b + c, a, b, c)$, where $t_{n-1} < a + b + c \leq t_n$. We use induction on n. We can check the first 'few' cases (by computer) and see that the theorem is true for $n = 2, 3, \ldots, 9$. (That is, $|S_1| \leq 81$.) Now, assume the result is true for all S_1 such that $|S_1| \leq t_k$, where $k < n$, $n \geq 10$.

If $S_1, \ldots, S_{10}$ are all monotonic, then, by the induction hypothesis and the lemma, either

$$L(S_1) = L(S_4) + 3 \leq 3\left[\frac{n-2}{2}\right] + 3 = 3\left[\frac{n}{2}\right]$$

or

$$L(S_1) = L(S_7) + 6 \leq 3\left[\frac{n-4}{2}\right] + 6 = 3\left[\frac{n}{2}\right]$$

or

$$L(S_1) = L(S_{10}) + 9 \leq 3\left[\frac{n-6}{2}\right] + 9 = 3\left[\frac{n}{2}\right].$$

Here we have used the fact that 2^t divides every element of S_{3t+1}, $t \geq 1$. Thus, for example, $S_4 = 2S_4^*$ and $L(S_4) = L(S_4^*)$. If $|S_4| \leq 2t_{n-2}$, then $|S_4^*| \leq t_{n-2}$, and so

$$L(S_4^*) \leq L(T_{n-2}) = 3\left[\frac{n-2}{2}\right],$$

by the induction hypothesis, taking S_4^* as our 'new' S_1. Thus, in any case,

$$L(S_1) \leq 3\left[\frac{n}{2}\right].$$

If $S_1, \ldots, S_{10}$ are not all monotonic, let S_j be the first which is nonmonotonic. Then

$$L(S_1) = L(S_j) + (j - 1) \leq 6 + j - 1 = j + 5 \leq 15,$$

since $L(S_j) \leq 6$ whenever S_j is not monotonic. But since $n \geq 10$,

$$L(T_n) = 3\left[\frac{n}{2}\right] \geq 15,$$

so $L(S_1) \leq L(T_n)$.

This completes the proof of Theorem 2.

Theorem 1 is now an easy corollary, since: if S_0 is monotonic decreasing and $|S_0| \leq |T_n|$, then $|S_1| \leq |T_n|$ and S_1 is additive. If S_0 is not monotonic decreasing, then $L(S_0) \leq 6$.

There actually are examples where $L(S) = L(T_n) + 1$:

$$L(T_6) = L(13, 7, 4, 2) = 9 \quad \text{and} \quad L(13, 6, 2, 0) = 10.$$

ly,

$$L(a + b + c,\ b + c,\ c,\ 0) = L(a + b + c,\ a,\ b,\ c) + 1;$$

$$L(t_n,\ t_{n-2} + t_{n-3},\ t_{n-3},\ 0) = L(T_n) + 1 = 3\left[\frac{n}{2}\right] + 1.$$

If we begin with a k-tuple of nonnegative integers, then it is known that $S_n = (0, 0, \ldots, 0)$ for sufficiently large n, provided $k = 2^t$. (If $k \neq 2^t$, the sequence $\{S_n\}$ may cycle [3], [4], [9].) Thus, a natural question to ask is: "What is the maximum length of the eight number game, or, more generally, the 2^t-number game?"

It was already mentioned that if S_1 is additive and leads to a long four-number game, then the ratios of the elements of S_1 should be close to the number $r = 1.839\ldots$. How accurately can the length of the game be predicted if one knows these ratios?

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