ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, PA 17745

Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to: RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extensions of old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.

PROBLEMS

H-339 *Proposed by Charles R. Wall, Trident Technical College, Charleston, CA*

A dyadic rational is a proper fraction whose denominator is a power of 2. Prove that 1/4 and 3/4 are the *only* dyadic rationals in the classical Cantor ternary set of numbers representable in base three using only 0 and 2 as digits.

H-340 *Proposed by Verner E. Hoggatt, Jr. (Deceased.)*

Let $A_2 = B$, $A_4 = C$, and $A_{2n+4} = A_{2n} - A_{2n+2}$ $(n = 1, 2, 3, \ldots)$. Show:

a. $A_{2n} = (-1)^{n+1}(F_{n-2}B - F_{n-1}C)$.

b. If $A_{2n} > 0$ for all $n > 0$, then $B/C = (1 + \sqrt{5})/2$.

H-341 *Proposed by Paul S. Bruckman, Corcord, CA*

Find the real roots, in exact radicals, of the polynomial equation

(1) $$p(x) \equiv x^6 - 4x^5 + 7x^4 - 9x^3 + 7x^2 - 4x + 1 = 0.$$

SOLUTIONS

Once Again

Professor M. S. Klamkin has pointed out that this problem was proposed previously by him (*Amer. Math. Monthly* 59 (1952):471]. It also appears in an article by W. E. Briggs, S. Chowla, A. J. Kempner, and W. E. Mientka entitled "On Some Infinite Series," *Scripta Math.* 21 (1955):28-30.

H-320 *Proposed by Paul S. Bruckman, Concord CA*
(Vol. 18, No. 4, December 1980)

Let

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},\ \text{Re}(s) > 1, \text{ the Riemann Zeta function.}$$

Also, let

$$H_n = \sum_{k=1}^{n} k^{-1},\ n = 1, 2, 3, \ldots, \text{ the harmonic sequence.}$$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3).$$

Solution by C. Georghiou, University of Patras, Patras, Greece

Method I: Clearly, the series converges; let S denote its sum. We note that

$$H_n = \sum_{k=1}^{n} k^{-1} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+n}\right) = \sum_{k=1}^{\infty} \frac{n}{k(k+n)} = -n\int_0^1 t^{n-1}\log(1-t)\,dy$$

and

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{x^{s-1}e^{-x}dx}{1-e^{-x}},\ \text{Re}(s) > 1,$$

where $\Gamma(s)$ is the Gamma function.

Then

$$S = \sum_{n=1}^{\infty} \frac{H_n}{n^2} = -\sum_{n=1}^{\infty}\int_0^1 \frac{t^n}{n}\,\frac{\log(1-t)}{t}dt = -\int_0^1 \sum_{n=1}^{\infty}\left(\frac{t^n}{n}\right)\frac{\log(1-t)}{t}dt$$

$$= \int_0^1 \frac{\log^2(1-t)}{t}\,dt, \text{ since } \sum_{n=1}^{\infty}\frac{t^n}{n} = -\log(1-t) \text{ for } |t| < 1,$$

and the interchange of the summation and integration signs is permissible.

Setting $t = 1 - e^{-x}$ in the last integral, we get

$$S = \int_0^{\infty}\frac{x^2e^{-x}dx}{1-e^{-x}} = 2\zeta(3).$$

Method II: Clearly, the series converges; let S denote its sum. We note that

$$H_n = \sum_{k=1}^{n} k^{-1} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+n}\right) = \sum_{k=1}^{\infty} \frac{n}{k(k+n)}.$$

Define also

$$H_n^2 = \sum_{k=1}^{n} k^{-2} \quad \text{and} \quad \overline{H}_n^2 = \zeta(2) - H_n^2.$$

Then

$\overline{H}_n^2 = \sum_{k=1}^{\infty} \frac{1}{(k+n)^2}$, the series $\sum_{n=1}^{\infty}\frac{\overline{H}_n^2}{n}$ converges, and we denote its sum by S.

The following hold true:

(i) $$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \overline{S}.$$

Indeed, by rearranging the above series, we get

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+k)^2} = \sum_{k=1}^{\infty} \frac{\overline{H}_k^2}{k} = \overline{S};$$

(ii) $$S = \zeta(3) + \overline{S}.$$

Indeed, with $H_0 = 0$,

$$S = \sum_{n=1}^{\infty} \frac{\frac{1}{n} + H_{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{H_{n-1}}{n^2} = \zeta(3) + \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2},$$

which, by means of (i) establishes (ii).

(iii) $$2\overline{S} = S.$$

Indeed,

$$\overline{S} = \sum_{n=1}^{\infty} \frac{\overline{H}_n^2}{n} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n(n+k)^2} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{1}{k^2}\left(\frac{k}{n(n+k)}\right) - \frac{1}{k}\left(\frac{1}{(k+n)^2}\right) \right\}$$
$$= \sum_{k=1}^{\infty} \frac{H_k}{k^2} - \sum_{k=1}^{\infty} \frac{\overline{H}_k^2}{k} = S - \overline{S},$$

from which (iii) follows.

Combining (ii) and (iii), we have $S = 2\zeta(3)$.

Also solved by L. Carlitz, A. G. Shannon, and the proposer.

Big Deal

H-321 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA (Vol. 18, No. 4, December 1980)*

Establish the identity

$$F_{n+14r}^6 + F_n^6 - (L_{12r} + L_{8r} + L_{4r} - 1)(F_{n+12r}^6 + F_{n+2r}^6)$$
$$+ (L_{20r} + L_{16r} + L_{4r} + 3)(F_{n+10r}^6 + F_{n+4r}^6)$$
$$- (L_{24r} - L_{20r} + L_{12r} + 2L_{8r} - 1)(F_{n+8r}^6 + F_{n+6r}^6)$$
$$= 40(-1)^n \prod_{i=1}^{3} F_{2ri}^2.$$

Solution by Paul S. Bruckman, Concord, CA

Let

(1) $$\phi(n, r) \equiv F^6_{n+14r} + F^6_n - A_r(F^6_{n+12r} + F^6_{n+2r}) + B\ (F^6_{n+10r} + F^6_{n+4r}) - C_r(F^6_{n+8r} + F^6_{n+6r}),$$

where

(2) $$A_r = L_{12r} + L_{8r} + L_{4r} - 1;$$

(3) $$B_r = L_{20r} + L_{16r} + L_{4r} + 3;$$

(4) $$C_r = L_{24r} - L_{20r} + L_{12r} + 2L_{8r} - 1.$$

We make repeated use of the following identities:

(5) $$L_{2u}L_{2v} = L_{2u+2v} + L_{2u-2v};$$

(6) $$F^6_m = 5^{-3}\{L_{6m} - 6(-1)^m L_{4m} + 15L_{2m} - 20(-1)^m\}.$$

It is a tedious but straightforward exercise to prove the following identities, by means of (5):

(7) $$L_{14kr} - A_rL_{10kr} + B_rL_{6kr} - C_rL_{2kr} = 0,\ k = 1, 2, 3.$$

Let

(8) $$U_r = C_r - B_r + A_r - 1.$$

We note that

$$\begin{aligned}125F^2_{2r}F^2_{4r}F^2_{6r} &= (L_{4r} - 2)(L_{8r} - 2)(L_{12r} - 2)\\ &= (L_{4r} - 2)(L_{20r} + L_{4r} - 2L_{12r} - 2L_{8r} + 4)\\ &= L_{24r} - 2L_{20r} - L_{16r} + 2L_{12r} + 3L_{8r} - 6\end{aligned}$$

(after simplification)

$$\begin{aligned}&= (L_{24r} - L_{20r} + L_{12r} + 2L_{8r} - 1) - (L_{20r} + L_{16r} + L_{4r} + 3)\\ &\quad + (L_{12r} + L_{8r} + L_{4r} - 1) - 1\\ &= C_r - B_r + A_r - 1,\end{aligned}$$

or

(9) $$U_r = 125F^2_{2r}F^2_{4r}F^2_{6r}.$$

Now

$$\begin{aligned}125\phi(n, r) = {}& L_{6n+84r} + L_{6n} - 6(-1)^nL_{4n+56r} - 6(-1)^nL_{4n} + 15L_{2n+28r} + 15L_{2n}\\ &- 40(-1)^n\\ &- A_r\{L_{6n+72r} + L_{6n+12r} - 6(-1)^nL_{4n+48r} - 6(-1)^nL_{4n+8r}\\ &\quad + 15L_{2n+24r} + 15L_{2n+4r} - 40(-1)^n\}\\ &+ B_r\{L_{6n+60r} + L_{6n+24r} - 6(-1)^nL_{4n+40r} - 6(-1)^nL_{4n+16r}\\ &\quad + 15L_{2n+20r} + 15L_{2n+8r} - 40(-1)^n\}\end{aligned}$$

$$- C_r\{L_{6n+48r} + L_{6n+36r} - 6(-1)^n L_{4n+32r} - 6(-1)^n L_{4n+24r} + 15L_{2n+16r} + 15L_{2n+12r} - 40(-1)^n\}$$

[using (6)]

$$= L_{6n+42r}(L_{42r} - A_r L_{30r} + B_r L_{18r} - C_r L_{6r}) - 6(-1)^n L_{4n+28r}(L_{28r} - A_r L_{20r} + B_r L_{12r} - C_r L_{4r}) + 15L_{2n+14r}(L_{14r} - A_r L_{10r} + B_r L_{6r} - C_r L_{2r}) + 40(-1)^n(-1 + A_r - B_r + C_r)$$

[using (5) repeatedly once again, and factoring]

$= 40(-1)^n U_r$ [using (7) and (8)], or as a result of (9),

(10) $$\phi(n, r) = 40(-1)^n F_{2r}^2 F_{4r}^2 F_{6r}^2. \quad \text{Q.E.D.}$$

Also solved by the proposer.

Two Much

H-322 *Proposed by Andreas N. Philippou, American Univ. of Beirut, Lebanon (Vol. 19, No. 1, February 1981)*

For each fixed integer $k \geq 2$, define the k-Fibonacci sequence $f_n^{(k)}$ by

$$f_0^{(k)} = 0,\ f_1^{(k)} = 1,$$

and

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)} & \text{if } 2 \leq n \leq k, \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} & \text{if } n \geq k+1. \end{cases}$$

Show the following:

(a) $f_n^{(k)} = 2^{n-2}$ if $2 \leq n \leq k+1$;

(b) $f_n^{(k)} < 2^{n-2}$ if $n \geq k+2$;

(c) $\sum_{n=1}^{\infty} (f_n^{(k)}/2^n) = 2^{k-1}$.

Solution by the proposer.

For $2 \leq n \leq k$,

$$f_n^{(k)} = 2^{n-2} f_2^{(k)} = 2^{n-2} \quad \text{and} \quad f_{(k+1)}^{(k+1)} = f_k^{(k)} + \cdots + f_1^{(k)} = 2f_k^{(k)} = 2^{k-1},$$

which establish (a). Next, for $n \geq k+1$,

$$f_n^{(k)} = f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} \quad \text{and} \quad f_{n-1}^{(k)} = f_{n-2}^{(k)} + \cdots + f_{n-1-k}^{(k)},$$

so that

(1) $$f_n^{(k)} = 2f_{n-1}^{(k)} - f_{n-1-k}^{(k)} \quad (n \geq k+1).$$

Taking $n = k + 2$ in (1), (a) implies $f_{k+2}^{(k)} = 2^k - 1 < 2^k$, which verifies (b) for $n = k + 2$. Assume now $f_m^{(k)} < 2^m$ for some integer m ($\geq k + 3$). It then follows by (1) and the positivity of $f_i^{(k)}$ $(i \geq 1)$, that

$$f_{m+1}^{(k)} = 2f_m^{(k)} - f_{m-k}^{(k)} < 2^m,$$

and this proves (b). Using (1) again, we get

$$(f_n^{(k)}/2^n) - (f_{n+1}^{(k)}/2^{n+1}) = (f_{n-k}^{(k)}/2^{n+1}) > 0 \qquad (n \geq k+1).$$

Therefore,

$$\lim_{n\to\infty} (f_n^{(k)}/2^n) = 0. \tag{2}$$

Setting $s_m^{(k)} = \sum_{n=1}^{m} (f_n^{(k)}/2^n)$ $(m \geq 1)$, and using (1) and (a), we get, after some algebra,

$$s_m^{(k)} = 2^{k-1} - (2^{k+1} - 1)(f_m^{(k)}/2^m) + \sum_{i=1}^{k} (f_{m-i}^{(k)}/2^{m-i}) \qquad (m \geq k+2). \tag{3}$$

Relations (2) and (3) give $\lim_{m\to\infty} s_m^{(k)} = 2^{k-1}$, and this shows (c).

Remark 1: For $k = 2$, (b) reduces to $F_n < 2^{n-2}$ if $n \geq 4$. (Fuchs [2] proposed and Scott [4] proved $F_n < 2^{n-2}$ if $n \geq 5$).

Remark 2: For $k = 2$, (c) reduces to $\sum_{n=1}^{\infty}(F_n/2^n) = 2$, a result obtained by Lind [3] in order to solve a problem of Brown [1].

References

1. J. L. Brown. Problem B-118. *The Fibonacci Quarterly* 5, no. 3 (1967):287.
2. J. A. Fuchs. Problem B-39. *The Fibonacci Quarterly* 2, no. 2 (1964):154.
3. D. Lind. Solution of Problem B-118. *The Fibonacci Quarterly* 6, no. 2 (1968):186.
4. B. Scott. Solution of Problem B-39. *The Fibonacci Quarterly* 2, no. 3 (1964):327.

Also solved by P. Bruckman and L. Somer.

A Common Recurrence

H-323 *Proposed by Paul Bruckman, Concord, CA*
(Vol. 19, No. 1, February 1981)

Let $(x_n)_0^\infty$ and $(y_n)_0^\infty$ be two sequences satisfying the common recurrence

$$p(E)z_n = 0, \tag{1}$$

where p is a monic polynomial of degree 2, and $E = 1 + \Delta$ is the unit right-shift operator of finite difference theory. Show that

$$x_n y_{n+1} - x_{n+1} y_n = (p(0))^n (x_0 y_1 - x_1 y_0), \; n = 0, 1, 2, \ldots . \tag{2}$$

Generalize to the case where p is of degree $e \geq 1$.

Solution by the proposer.

We solve the general case, with p any monic polynomial of degree $e \geq 1$. Suppose

$$(z_n^{(1)})_0^\infty, \; (z_n^{(2)})_0^\infty, \; \ldots, \; (z_n^{(e)})_0^\infty$$

are sequences satisfying the common recursion (1). We seek to evaluate Casorati's Determinant

(3) $$D_n = \begin{vmatrix} z_n^{(1)} & z_n^{(2)} & \cdots & z_n^{(e)} \\ z_{n+1}^{(1)} & z_{n+1}^{(2)} & \cdots & z_{n+1}^{(e)} \\ \vdots & \vdots & & \vdots \\ z_{n+e-1}^{(1)} & z_{n+e-1}^{(2)} & \cdots & z_{n+e-1}^{(e)} \end{vmatrix}$$

Let $U_n = ((z_{n+i-1}^{(j)}))_{e \times e}$ be the matrix whose determinant is D_n; also, define the $e \times e$ matrix J as follows:

(4) $$J = \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -p(0) & -p'(0)/1! & -p''(0)/2! & -p'''(0)/3! & \cdots & -p^{(e-1)}(0)/(e-1)! \end{vmatrix}.$$

Note that p has the Maclaurin Series expansion

(5) $$p(z) = \sum_{r=0}^{e} \frac{p^{(r)}(0)}{r!} z^r.$$

Therefore, the sequences $(z_n^{(k)})_0^\infty$ $(k = 1, 2, \ldots, e)$ satisfy the common recursion

$$p(E)z_n = \sum_{r=0}^{e} \frac{p^{(r)}(0)}{r!} E^r z_n = \sum_{r=0}^{e} \frac{p^{(r)}(0)}{r!} z_{n+r} = 0;$$

since p is monic, $p^{(e)}(0)/e! = 1$, and hence

(6) $$z_{n+e} = -\sum_{r=0}^{e-1} \frac{p^{(r)}(0)}{r!} z_{n+r} \quad (n = 0, 1, 2, \ldots).$$

We now observe, using (3), (4), and (6), that

(7) $$J \cdot U_n = U_{n+1}, \; n = 0, 1, 2, \ldots .$$

It follows by an easy induction that

(8) $$U_n = J^n U_0, \; n = 0, 1, 2, \ldots .$$

We may evaluate $|J|$ along the first column of J, and we find readily that $|J| = (-1)^{e-1}(-p(0)) = (-1)^e p(0)$. Therefore, taking determinants in (8) yields:

$$D_n = |U_n| = |J^n| \cdot |U_0| = |J|^n \cdot D_0,$$

or

(9) $$D_n = \{(-1)^e p(0)\}^n D_0, \; n = 0, 1, 2, \ldots .$$

This is the desired generalization of (2). Many interesting identities arise by specializing further. For example, taking

$$p(z) = z^2 - z - 1, \; (x_n) = (F_n), \text{ and } (y_n) = (L_n),$$

yields:

(10) $$F_n L_{n+1} - F_{n+1} L_n = 2(-1)^{n-1}, \; n = 0, 1, 2, \ldots .$$

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ERRATA

In the article "On the Fibonacci Numbers Minus One" by G. Geldenhuys, Volume 19, no. 5, the following two errors appear on pages 456 and 457:

1. The recurrence relation (1), which appears as

$$D_1 = 1 + \mu, \; D_2 = (1 - \mu)^2, \text{ and } D_n = (1 + \mu)D_{n-1} - \mu D_{n-3} \text{ for } n \geq 3$$

should read

$$D_1 = 1 + \mu, \; D_2 = (1 + \mu)^2, \text{ and } D_n = (1 + \mu)D_{n-1} - \mu D_{n-3} \text{ for } n \geq 3;$$

2. The alternative recurrence relation (4), which appears as

$$D_m - D_{m-1} - D_{m-2} = 1 \text{ for } m \geq 3$$

should read

$$D_m - \mu D_{m-1} - \mu D_{m-2} = 1 \text{ for } m \geq 3.$$

We thank Professor Geldenhuys for bringing this to our attention.

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