GEOMETRY OF A GENERALIZED SIMSON'S FORMULA

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(Submitted April 1981)

1. Introduction

Articles of a geometrical nature relating to recurrence sequences have appeared in recent years in this journal (e.g. [1], [2], [6]).

The purpose of the present paper is to consider the loci in the Euclidean plane satisfied by points whose Cartesian coordinates are pairs of successive numbers in recurrence sequences of a certain type. Readers might plot some points on the resulting noncontinuous curves (conics).

Extension to higher-dimensional space is briefly discussed.

2. The General Conic

Begin by defining [4] the general term of the sequence $\{w_n(a, b; p, q)\}$ as

(1) $$w_{n+2} = pw_{n+1} - qw_n, \; w_0 = a, \; w_1 = b,$$

where a, b, p, and q belong to some number system, but are usually thought of as integers. Write [4]

(2) $$e = pab - qa^2 - b^2.$$

Now [4]

(3) $$w_n w_{n+2} - w_{n+1}^2 = eq^n,$$

which is a generalization of Simson's formula

(4) $$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}$$

occurring in the Fibonacci sequence $\{F_n\} = \{w_n(0, 1; 1, -1)\}$.

Equation (3) generalizes the famous geometrical paradox associated with (4). For the details, see [5].

From (1) and (3), we obtain

(5) $$qw_n^2 + w_{n+1}^2 - pw_n w_{n+1} + eq^n = 0.$$

Next, put $w_n = x$, $w_{n+1} = y$. Then, by (5),

(6) $$qx^2 + y^2 - pxy + eq^n = 0.$$

This equation represents a conic in rectangular Cartesian coordinates (x, y). Anticlockwise rotation of axes through an angle

$$\frac{1}{2} \tan^{-1}\left(\frac{-p}{q - 1}\right)$$

eliminates the xy term and produces the canonical form of the conic (an ellipse if $p^2 < 4q$, a hyperbola if $p^2 > 4q$, where the degenerate cases are excluded). Equation (6) is also obtainable by laborious reduction of the general equation of a conic using the uniqueness of a conic through 5 given points.

3. Some Particular Cases

I. $q < 0$ (Hyperbolas)

(a) $p = 1, q = -1$: Substituting in (6) yields the two systems (n even, n odd) of rectangular hyperbolas

$$x^2 - y^2 + xy = e_1(-1)^n \qquad (e_1 = a^2 - b^2 + ab), \tag{7}$$

asymptotes of which are the perpendicular lines

$$y = \alpha x, \; y = -\frac{1}{\alpha}x, \tag{8}$$

in which $\alpha = \dfrac{1 + \sqrt{5}}{2}$, the positive root of $t^2 - t - 1 = 0$. For the *Fibonacci sequence* ($a = 0$, $b = 1$) and the *Lucas sequence* ($a = 2$, $b = 1$), it follows that $e_1 = -1$ and 5, respectively. These *Fibonacci-type* curves (7) approach their asymptotes remarkably quickly.

With a fixed e_1 in (7), a hyperbola for which n is odd (even) may be transformed into the corresponding hyperbola for which n is even (odd), by a reflection in $y = x$ followed by a reflection in the y-axis (x-axis).

(b) $p = 2, q = -1$: For the *Pell sequence* ($a = 0$, $b = 1$), (6) gives

$$x^2 - y^2 + 2xy = (-1)^{n+1}, \tag{9}$$

rectangular hyperbolas with perpendicular asymptotes $y = kx$, $y = -\frac{1}{k}x$, where $k = 1 + \sqrt{2}$ is the positive root of $t^2 - 2t - 1 = 0$.

Gradients of the perpendicular asymptotes of the hyperbolas (6) for which $p > 0$, $q = -1$ are given by the roots of $t^2 - pt - 1 = 0$.

II. $q > 0$

Equation (6) now represents ellipses if $4q > p^2$ and hyperbolas if $4q < p^2$. For example, the loci for the *Fermat sequences*

$$\{w_n(0,\ 1;\ 3,\ 2)\} \quad \text{and} \quad \left\{w_n\left(\frac{3}{2},\ 2;\ 3,\ 2\right)\right\}$$

are hyperbolas (one point for each n)

$$2x^2 + y^2 - 3xy = 2^n \tag{10}$$

and

$$2x^2 + y^2 - 3xy = -2^{n-1}. \tag{11}$$

Further, for the *Chebyshev sequences*

$$\{w_n(1,\ 2\lambda;\ 2\lambda,\ 1)\} \quad \text{and} \quad \{w_n(2,\ 2\lambda;\ 2\lambda,\ 1)\},$$

where $\lambda = \cos\theta$, we obtain the ellipses

$$x^2 + y^2 - 2\lambda xy = 1 \tag{12}$$

and

$$x^2 + y^2 - 2\lambda xy = 4 - 4\lambda^2. \tag{13}$$

III. Degenerate Case

When $\lambda = 1$ in (12), i.e., for the sequence $\{w\ (1,\ 2;\ 2,\ 1)\}$, of *integers*, we have the degenerate curve $x^2 + y^2 - 2xy = 1$, i.e., the line

$$x - y = -1.$$

No values of $x + y$, as defined, satisfy the equation $x - y = 1$. Successive pairs of *odd integers* and of *even integers*, generated by

$$\{w_n(1,\ 3;\ 2,\ 1)\} \quad \text{and} \quad \{w_n(2,\ 4;\ 2,\ 1)\},$$

respectively, satisfy the line

$$x - y = -2. \tag{15}$$

4. Extension to Higher Space

Equations of the third, fourth, and higher degrees that are based on second-order recurrences like (1) (see, e.g. [3], [4]) cannot yield any nondegenerate loci in spaces of dimension greater than two.

For three-dimensional (nonprojective) space, it is necessary to consider third-order recurrence relations, of which the simplest is

$$P_{n+3} = P_{n+2} + P_{n+1} + P_n \qquad (n \geq 0). \tag{16}$$

Waddill and Sacks [8] have established the following relation for $\{P_n\}$ corresponding to the Simson formula (4) for $\{F_n\}$:

$$\begin{aligned} &P^2_{n+3}P_n + P^3_{n+2} + P^2_{n+1}P_{n+4} - P_{n+4}P_{n+2}P_n - 2P_{n+3}P_{n+2}P_{n+1} \\ &= P^3_0 + 2P^3_1 + P^3_2 + 2P^2_0P_1 + 2P_0P^2_1 + P^2_0P_2 - 2P_1P^2_2 - 2P_0P_1P_2 - P_0P^2_2. \end{aligned} \tag{17}$$

Putting $P_0 = 0$, $P_1 = P_2 = 1$ and $P_0 = 1, P_1 = 0, P_2 = 1$ they obtained their sequences $\{K_n\}$ and $\{Q_n\}$, respectively:

(18) $\{K_n\}$: 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, ...;

(19) $\{Q_n\}$: 1, 0, 1, 2, 3, 6, 11, 20, 37, 68, 125, 230, ...

Letting $P_n = x$, $P_{n+1} = y$, $P_{n+2} = z$ in (17), we derive, after some algebraic manipulation,

$$x^3 + 2y^3 + z^3 + 2x^2y + 2xy^2 - 2yz^2 + x^2z - xz^2 - 2xyz = A, \tag{20}$$

where $A = 1$ for $\{K_n\}$ and $A = 2$ for $\{Q_n\}$. Equations (20) represent cubic surfaces in Euclidean space of three dimensions.

More general forms of (16) would lead to extremely cumbersome equations.

Observe that if we label any three successive numbers in (18) as x, y, z, and the corresponding three numbers in (19) as X, Y, Z, then we perceive that $X = y - x$, $Y = z - y$, $Z = x + y$.

Fourth- and higher-order recurrences should produce equations corresponding to (17) which are generalizations of Simson's formula (4). While (17) is not a pretty sight, the mind boggles at the prospect of further extensions, which we accordingly do not investigate. But the general pattern seems clear: a recurrence of the nth order ought to lead to a hypersurface (of dimension $n - 1$) in Euclidean n-space.

5. Concluding Comments

a. For the sequence $\{w_n(1, \alpha; 1, -1)\}$, $e = 0$ [see (1), (2), (7)] and the curve (7) degenerates to the line-pair $x^2 + xy - y^2 = 0$.

b. Graphing the Fibonacci numbers F_n against n reveals that they asymptotically approach the exponential values

$$\lim_{n\to\infty} F_n = \frac{\alpha^n}{\sqrt{5}} \qquad \left(\text{where } F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},\ \beta = -\frac{1}{\alpha}\right).$$

c. M. H. Eggar, in "Applications of Fibonacci Numbers" [*The Mathematical Gazette* 63 (1979):36-39], refers to (7), in the case where $e_1 = 1$, though his context is nongeometrical.

d. Interest in the theme of this article was stimulated by a private communication to the author in 1980 by L. G. Wilson, who determined the vertex of the hyperbola (7) for the Fibonacci sequence, but only in the case where n is odd, namely, $x = 0.920442065\ldots$, $y = 0.217286896\ldots$. He also calculated the angle of inclination of the axis of this hyperbola to the x-axis, namely,

$$13.28252259\ldots \text{ degrees } [(\doteqdot 13°17') = \tan^{-1}(\sqrt{5} - 2)].$$

Furthermore, Wilson briefly investigated the geometry of the third-order sequence $\{T_n\}$:

$$0,\ 2,\ 3,\ 6,\ 10,\ 20,\ 35,\ 66,\ \ldots, \tag{21}$$

defined in Neumann-Wilson [7] by

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n + (-1)^n \qquad (n \geq 0). \tag{22}$$

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AN ENTROPY VIEW OF FIBONACCI TREES*

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(Submitted May 1981)

Abstract

In a binary tree with n terminal nodes weighted by probabilities $p_1, \ldots, p_n$, $\Sigma p_i = 1$, it is assumed that each left branch has cost 1 and each right branch has cost 2. The cost a_i of terminal node p_i is defined to be the sum of costs of branches that form the path from the root to this node. The sum $\Sigma p_i a_i$ is called the average cost of the tree. As a top-down tree-building rule we consider *ψ-weight-balancing* which constructs a binary tree by successive dichotomies of the ordered set $p_1, \ldots, p_n$ according to a certain weight ratio closely approximating the golden ratio. Let $H = H(p_1, \ldots, p_n) = -\Sigma p_i \log p_i$ be the Shannon entropy of these probabilities. The ψ-weight-balancing rule is motivated by the fact that the entropy per unit of cost

$$H(x, 1 - x)/(1 \cdot x + 2 \cdot (1 - x))$$

for the division $x : (1 - x)$ of the unit interval is maximized when

$$x = \psi = (\sqrt{5} - 1)/2,$$

the golden cut point. It is then shown that the average cost of the tree built by ψ-weight-balancing is bounded above by $H/(-\log \psi) + 1$, if the terminal nodes have probabilities $p_1, \ldots, p_n$, $p_1 \geq \cdots \geq p_n$, from left to right in this order in the tree. If $p_{j+1}/p_j \geq (1/2)\psi$ for each j, the above bound can be improved to $H/(-\log \psi) + \psi$. For the case $p_1 = \cdots = p_n$, we obtain the following results. The ψ-weight-balancing constructs an optimal tree in the sense of minimum average cost and constructs the Fibonacci tree of order k when $n = F_k$, the kth Fibonacci number. The average cost of the optimal tree is given exactly. Furthermore, for an arbitrarily given number of terminal nodes, the ψ-weight-balanced tree is also "balanced" in the sense of Adelson-Velskii and Landis, and is the highest of all balanced trees.

We will discuss some properties of Fibonacci (Fibonaccian) trees in view of their construction by an entropic weight-balancing, beginning with the following preparatory section:

**This paper was presented at the International Colloquium on Information Theory, Budapest, Hungary, August 24-28, 1981.*