NOTES ON SUMS OF PRODUCTS OF GENERALIZED FIBONACCI NUMBERS

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The infinite sequence $\{s_n\}$ is a sequence of *generalized Fibonacci numbers* (also called a *generalized Fibonacci sequence* or simply *Fibonacci sequence*) if $s = s_{n-1} + s_{n-2}$ for all n. A particular Fibonacci sequence is completely specified by any two consecutive terms. In this paper we let $\{f_n\}$, $\{g_n\}$, $\{h_n\}$, and $\{k_n\}$ represent generalized Fibonacci sequences, and we let $\{F_n\}$ represent the sequence of *Fibonacci numbers* defined by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

Theorem: If $\{f_n\}$, $\{g_n\}$, and $\{h_n\}$ are Fibonacci sequences, then the following summations hold for $y \geq x$.

(1) $$\sum_{x \leq i \leq y} f_{i+r} = [f_{n+r+2}]_{n=x-1}^{n=y}$$

(2) $$2\sum_{x \leq i \leq y} f_{i+r}g_{i+s} = [f_{n+r}g_{n+s+1} + f_{n+r+1}g_{n+s}]_{n=x-1}^{n=y}$$

(3) $$\begin{aligned}2\sum_{x \leq i \leq y} f_{i+r}g_{i+s}h_{i+t} = [&f_{n+r+1}g_{n+s+1}h_{n+t} + f_{n+r+1}g_{n+s}h_{n+t+1} \\ &+ f_{n+r}g_{n+s+1}h_{n+t+1} - f_{n+r+1}g_{n+s+1}h_{n+t+1} \\ &- f_{n+r}g_{n+s}h_{n+t}]_{n=x-1}^{n=y}.\end{aligned}$$

Proof: The proofs are by induction on y. As base cases we take $y = x - 1$; the summations are empty, and the right-hand sides vanish identically. The induction steps are as follows:

1. $$\begin{aligned}\sum_{x \leq i \leq y+1} f_{i+r} &= \sum_{x \leq i \leq y} f_{i+r} + f_{y+r+1} \\ &= [f_{n+r+2}]_{n=x-1}^{n=y} + f_{y+r+1} \\ &= [f_{y+r+2} + f_{y+r+1}] - [f_{(x-1)+r+2}] \\ &= [f_{y+r+3}] - [f_{(x-1)+r+2}] \\ &= [f_{n+r+2}]_{n=x-1}^{n=y+1}.\end{aligned}$$

2. $$\begin{aligned}2\sum_{x \leq i \leq y+1} f_{i+r}g_{i+s} &= 2\sum_{x \leq i \leq y} f_{i+r}g_{i+s} + 2f_{y+r+1}g_{y+s+1} \\ &= [f_{n+r}g_{n+s+1} + f_{n+r+1}g_{n+s}]_{n=x-1}^{n=y} + 2f_{y+r+1}g_{y+s+1}\end{aligned}$$

(continued)

$$= [(f_{y+r}g_{y+s+1} + f_{y+r+1}g_{y+s+1}) + (f_{y+r+1}g_{y+s} + f_{y+r+1}g_{y+s+1})]$$
$$- [f_{(x-1)+r}g_{(x-1)+s+1} + f_{(x-1)+r+1}g_{(x-1)+s}]$$
$$= [f_{y+r+2}g_{y+s+1} + f_{y+r+1}g_{y+s+2}]$$
$$- [f_{(x-1)+r}g_{(x-1)+s+1} + f_{(x-1)+r+1}g_{(x-1)+s}]$$
$$= [f_{n+r}g_{n+s+1} + f_{n+r+1}g_{n+s}]_{n=x-1}^{n=y+1}.$$

3. We note that, as in the proofs of (1) and (2), the bottom limit of the right-hand side is unchanged, and we need only prove the following identity related to the top limit y:

$$(*)\quad [f_{y+r+1}g_{y+s+1}h_{y+t} + f_{y+r+1}g_{y+s}h_{y+t+1} + f_{y+r}g_{y+s+1}h_{y+t+1}$$
$$- f_{y+r+1}g_{y+s+1}h_{y+t+1} - f_{y+r}g_{y+s}h_{y+t}] + 2f_{y+r+1}g_{y+s+1}h_{y+t+1}$$
$$= [f_{y+r+2}g_{y+s+2}g_{y+t+1} + f_{y+r+2}g_{y+s+1}h_{y+t+2} + f_{y+r+1}g_{y+s+2}h_{y+t+2}$$
$$- f_{y+r+2}g_{y+s+2}h_{y+t+2} - f_{y+r+1}g_{y+s+1}h_{y+t+1}].$$

As a shorthand notation we let (abc) stand for $f_{y+r+a}g_{y+s+b}h_{y+t+c}$, where a, b, c are 0, 1, or 2. Then the identity (*) can be written as follows:

$$(**)\quad (110) + (101) + (011) - (111) - (000) + 2(111)$$
$$= (221) + (212) + (122) - (222) - (111).$$

The following identities are easily verified, and the validity of (**), and therefore of (3), follows immediately:

$$(221) = (001) + (011) + (101) + (111),$$

$$(212) = (010) + (110) + (011) + (111),$$

$$(122) = (100) + (110) + (101) + (111),$$

$$(222) = (000) + (001) + (010) + (011) + (100) + (101) + (110) + (111). \square$$

Identity (1) is well known, although it is usually stated in terms of Fibonacci or Lucas numbers with limits of summations 1 to n or 0 to n.

Identity (2) is a generalization of the identities of Berzsenyi [1]. This is easily shown using the following identity, which is easily verified, where $\{f_n\}$ and $\{g_n\}$ are generalized Fibonacci sequences:

$$(4)\quad f_{n+k}g_{m-k} - f_n g_m = (-1)^n (f_k g_{m-n-k} - f_0 g_{m-n}).$$

We have the following:

$$2\sum_{0 \le i \le n} f_i g_{i+2m+b} = [f_{i+1}g_{i+2m+b} + f_i g_{i+2m+b+1}]_{i=-1}^{i=n}$$

$$= [f_{m+b+i}g_{m+i+1} - (-1)^{i+1}(f_{m+b-1}g_m - f_0g_{2m+b-1})$$
$$+ f_{m+b+i}g_{m+i+1} - (-1)^i(f_{m+b}g_{m+1} - f_0g_{2m+b+1})]_{i=-1}^{i=n}$$
$$= [2f_{m+b+i}g_{m+i+1} + (-1)^i(f_{m+b-1}g_m - f_{m+b}g_{m+1} + f_0g_{2m+b})]_{i=-1}^{i=n}.$$

Applying the limits, we obtain the following expression:

$$\sum_{0\le i\le n} f_i g_{i+2m+b} = f_{n+m+b}g_{n+m+1} - f_{m+b-1}g_m + \left(\frac{1+(-1)^n}{2}\right)(f_{m+b-1}g_m - f_{m+b}g_{m+1} + f_0g_{2m+b}),$$

which is exactly the expression obtained in [1] for even or odd n and $b = 0$ or 1.

The advantages of identity (2), besides its attractive symmetry, are that (a) only a single case is needed instead of four separate cases and (b) it is applicable to general limits, not just the sum from 0 to n. In addition, (2) applies to the sum of the product of terms from *different* generalized Fibonacci sequences, as opposed to the original form in [1].

It should be noted that (4) can also be applied directly to (2), leading to the summation

$$\sum_{x\le i\le y} f_{i+r}g_{i+s} = \left[f_{n+r}g_{n+s+1} + \frac{(-1)^{n+r}}{2}(f_1g_{s-r} - f_0g_{s-r+1})\right]_{n=x-1}^{n=y}.$$

The summation of (2) has also been considered by Pond [1], whose result is valid for the sum of the products of terms from *identical* generalized Fibonacci sequences:

$$\sum_{x\le i\le y} f_i f_{i+s} = \left[\frac{1}{2}(F_{s-3}f_nf_{n+1} + F_sf_{n+2}^2)\right]_{n=x-1}^{n=y}.$$

Recall that $\{F_n\}$ is the sequence of Fibonacci numbers. This result is easily derived from (2) by use of the identity $f_{n+r} = F_{r-1}f_n + F_rf_{n+1}$.

Identity (3) has been considered by Pond [2], again in a simpler context; he requires all *three* generalized Fibonacci sequences to be identical and derives the following expression:

$$2\sum_{x\le i\le y} f_i f_{i+r} f_{i+s} = \left[(F_sF_r - F_{s-1}F_{r-1})D(-1)^n f_{n-1} + f_{s+r+n+1}f_nf_{n+1}\right]_{n=x-1}^{n=y},$$

where $D(-1)^n = f_{n-1}f_{n+1} - f_n^2 = (-1)^n(f_{-1}f_1 - f_0^2)$. It is not hard to show that this summation is a consequence of (3).

The advantages of identity (3) again lie in its pleasing symmetry, its applicability to general limits, and the fact that the summation is valid for products of *different* Fibonacci sequences.

A general methodology for finding summations of the form of identities (1), (2), or (3) has been discussed elsewhere [3]. This methodology expresses the sum of the products of terms from several sequences, each defined by a linear recurrence, as a *standard sum*, defined below.

If the sum to be found is

$$\sum_{x \le n \le y} f_{1,\, n+r_1} f_{2,\, n+r_2} \cdots f_{m,\, n+r_m}$$

where the m sequences $\{f_1\}$, ..., $\{f_m\}$ are defined by linear recurrence relations (i.e., not necessarily Fibonacci sequences), the standard sum is a linear combination

$$\left[\sum_{(i_1, \ldots, i_m) \in I^m} a_{i_1, i_2, \ldots, i_m} f_{1, n+r_1+i_1} f_{2, n+r_2+i_2} \cdots f_{m, n+r_m+i_m}\right]_{n=x-1}^{n=y}$$

with the following important properties:

1. each term of the standard sum is the product of m terms, one from each of the original sequences in the product to be summed;

2. the m-tuples $(i_1, \ldots, i_m)$ have constant integer components;

3. the coefficients $a_{i_1, \ldots, i_m}$ are constant and only a bounded number of the coefficients are nonzero.

Of interest is the result that a standard sum for the sum of the products of terms from recurrence sequences does not always exist. In particular, the sum

$$\sum f_{n+r} g_{n+s} h_{n+t} k_{n+u}$$

(with $\{f_n\}$, $\{g_n\}$, $\{h_n\}$, and $\{k_n\}$ Fibonacci sequences) cannot be expressed as a standard sum. For details, the interested reader is referred to [3].

References

1. G. Berzsenyi. "Sums of Products of Generalized Fibonacci Numbers." *The Fibonacci Quarterly* 13, no. 4 (1975):343-344.
2. J. C. Pond. "Generalized Fibonacci Summations." *The Fibonacci Quarterly* 6, no. 2 (1968):97-108.
3. D. L. Russell. "Sum of Products of Terms from Linear Recurrence Sequences." *Discrete Mathematics* 28 (1979):65-79.

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