ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be on a separate sheet (or sheets). Preference will be given to those that are typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-478 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

(a) Show that the congruence

$$x^2 \equiv -1 \pmod{4m^2 + 4m + 5}$$

has $x = \pm(2m^2 + m + 2)$ as a solution for m in $N = \{0, 1, \ldots\}$.

(b) Show that the congruence

$$x^2 \equiv -1 \pmod{100m^2 + 156m + 61}$$

has a solution $x = am^2 + bm + c$ with fixed integers a, b, c for m in N.

B-479 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $L_{a+nd} + L_{a+nd-d} - L_{a+d} - L_a$ is an integral multiple of L_d for positive integers a, d, n with d odd.

B-480 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $L_{a+nd} - L_{a+nd-d} - L_{a+d} + L_a$ is an integral multiple of $L_d - 2$ for positive integers a, d, n with d even.

B-481 *Proposed by Jerry Metzger, University of N. Dakota, Grand Forks, ND*

A and B compare pennies with A winning when there is a match. During an unusual sequence of mn comparisons, A produced m heads followed by m tails followed by m heads, etc., while B produced n heads followed by n tails followed by n heads, etc. By how much did A's wins exceed his losses? [For example, with $m = 3$ and $n = 5$, one has

A: HHHTTTHHHTTTHHH
B: HHHHHTTTTTHHHHH

and A's 8 wins exceeds his 7 losses by 1.]

B-482 *Proposed by John Hughes and Jeff Shallit, U.C., Berkeley, CA*

Find an infinite sequence $a_1, a_2, \ldots,$ of positive integers such that

$$\lim_{n\to\infty} (a_n)^{1/n} \text{ and } \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k)\right]$$

both exist and are unequal.

B-483 *Proposed by John Hughes and Jeff Shallit, U.C., Berkeley, CA*

Find an infinite sequence $a_1, a_2, \ldots,$ of positive integers such that

$$\lim_{n\to\infty} (a_n)^{1/n} \text{ exists and } \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k)\right] \text{ does not exist.}$$

SOLUTIONS

Generating F_n^2 and L_n^2

B-452 *Proposed by P. L. Mana, Albuquerque, NM*

Let $c_0 + c_1x + c_2x^2 + \cdots$ be the Maclaurin expansion for

$$[(1 - ax)(1 - bx)]^{-1},$$

where $a \neq b$. Find the rational function whose Maclaurin expansion is

$$c_0^2 + c_1^2x + c_2^2x^2 + \cdots$$

and use this to obtain the generating functions for F_n^2 and L_n^2.

Solution by A. G. Shannon, New South Wales Inst. of Tech., Sydney, Australia

Let $U(x) = \sum_{n=0}^{\infty} x^n$, formally. Then $\sum_{n=0}^{\infty} c_nx^n = [(1-ax)(1-bx)]^{-1}$ yields

$c_n = (a^{n+1} - b^{n+1})/(a - b)$ so that

$$\sum_{n=0}^{\infty} c_n^2 x^n = (a^2U(a^2x) + b^2U(b^2x) - 2abU(abx))/(a - b)^2$$
$$= (1 + abx)/(1 - a^2x)(1 - b^2x)(1 - abx).$$

Thus, when $a = 1 - b = \frac{1}{2}(1 + \sqrt{5})$, we have

$$\sum_{n=0}^{\infty} F_n^2 x^n = \sum_{n=1}^{\infty} F_n^2 x^n = x\sum_{n=1}^{\infty} c_{n-1}^2 x^{n-1} = x(1 - x)/(1 - a^2x)(1 - b^2x)(1 + x)$$
$$= (x - x^2)/(1 - 2x - 2x^2 + x^3).$$

Also

$$\sum_{n=0}^{\infty} L_n^2 x^n = U(a^2x) + U(b^2x) + 2U(abx) = (4 + 7x - x^2)/(1 - a^2x)(1 - b^2x)(1 + x)$$
$$= (4 - 7x - x^2)/(1 - 2x - 2x^2 + x^3).$$

These results are particular cases of Eqs. (33) and (42) of A. F. Horadam's article, "Generating Functions for Powers of a Certain Generalised Sequence of Numbers," *Duke Math. J.* 32 (1965):437-46.

Also solved by Paul S. Bruckman, John Ivie, E. Primrose, Heinz-Jürgen Seiffert, Sahib Singh, John Spraggan, Gregory Wulczyn, and the proposer.

FiFibonacci and LuLucas Equations

B-453 *Proposed by Paul S. Bruckman, Concord, CA*

Solve in integers r, s, t with $0 \leq r < s < t$ the FiFibonacci Diophantine equation

$$F_{F_r} + F_{F_s} = F_{F_t}$$

and the analogous LuLucas equation in which each F if replaced by an L.

Solution by Sahib Singh, Clarion State College, Clarion, PA

It is easy to see that the FiFibonacci Diophantine equation has solutions:

$r = 0;\ s = 1,\ 2;\ t = 3$
$r = 1;\ s = 2,\ 3;\ t = 4$
$r = 2,\ s = 3;\ \ t = 4$

and the LuLucas equation admits the following solutions:

$r = 0;\ s = 1;\ t = 2$
$r = 0;\ s = 2;\ t = 3$

We show that there is no other solution possible.

After considering the above values, we see that onward the values of F_r and F_s are neither equal nor consecutive. Hence $F_s - 1$ is greater than F_r. Thus $F_{F_s} < F_{F_r} + F_{F_s} < F_{(F_s - 1)} + F_{F_s} = F_{(F_s + 1)}$. Therefore, $F_{F_r} + F_{F_s}$ lies between two consecutive Fibonacci numbers F_{F_s} and $F_{(F_s + 1)}$ and cannot qualify as a Fibonacci number. Thus no other solution is possible. Similar arguments enable us to conclude that no other solution is possible for the LuLucas Diophantine equation.

Also solved by Herta T. Freitag, John Ivie, Lawrence Somer, and the proposer.

Magic Corners

B-454 *Proposed by Charles W. Trigg, San Diego, CA*

In the square array of the nine nonzero digits
$$\begin{matrix} 6 & 7 & 5 \\ 2 & 1 & 3 \\ 9 & 4 & 8 \end{matrix}$$
the sum of the four digits in each 2-by-2 corner array is 16. Rearrange the nine digits so that the sum of the digits in each such corner array is *seven* times the central digit.

Solution by J. Suck, Essen, E. Germany

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 9 & 2 \\ 8 & 3 & 7 \\ 4 & 6 & 5 \end{pmatrix}$$ is up to rotating and flipping the only solution:

$e = 1$ is impossible, since $9 > 7$ must be in some corner, $e = 2$ is impossible from the fact that, in the corner of the 9, the sum of the other two digits would have to be 3, $e \geq 4$ is impossible, since the sum of the remaining three digits in a corner is at most $7 + 8 + 9 = 24$ with no other corner reaching this.

Now, let $e = 3$. Then the sum of all corner sums is $4 \cdot 21 = 1 + 2 + \cdots + 9 + b + d + f + h + 3 \cdot 3$, i.e., $b + d + f + h = 30$, showing that
$$\{b, d, f, h\} = \{6, 7, 8, 9\}.$$
Let, say, $b = 9$, then $h = 8$ is impossible (since, say, $f = 7$ so that $i = 3$, and $d = 6$ so that $a = 3$, too). Also, $h = 7$ is impossible (since, say, $f = 6$ making $g = 3$, and $d = 8$ making $g = 3$, too). Thus, $h = 6$, which entails the given solution.

Also solved by Paul S. Bruckman, Karen S. Carter, Derek Chang, Frank Higgins, Walther Janous, Birgit Kober, John Milsom, Bob Prielipp, Sahib Singh, Lawrence Somer, W. R. Utz, Gregory Wulczyn, and the proposer.

Simplified Convolution

B-455 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_m = \sum_{i=0}^{m} F_{i+1}L_{m-i}$ and $T_m = 10S_m/(m+2)$. Prove that T_m is a sum of two Lucas numbers for $m = 0, 1, 2, \ldots$.

Solution by Sahib Singh, Clarion State College, Clarion, PA

$$S_m = \sum_{i=0}^{m} F_{i+1}L_{m-i} = \frac{1}{5} \sum_{i=0}^{m} (L_i + L_{i+2})L_{m-i}.$$
Using $L_i = a^i + b^i$, the above summation becomes
$$\begin{aligned} S_m &= \frac{1}{5} \sum_{i=0}^{m} [(L_m + L_{m+2}) + (-1)^i\{L_{m-2i-2} + L_{m-2i}\}] \\ &= \frac{1}{5}[(m + 1)(L_m + L_{m+2}) + (L_m + L_{m+2})] \\ &= \frac{(m + 2)}{5}(L_m + L_{m+2}). \end{aligned}$$

Thus,

$$T_m = 2(L_{m+2} + L_m) = L_{m+4} + L_{m-2}.$$

Also solved by Paul S. Bruckman, Frank Higgins, B. S. Popov, Bob Prielipp, Gregory Wulczyn, and the proposer.

Fibonacci Products of Two Primes

B-456 *Proposed by Albert A. Mullin, Huntsville, AL*

It is well known that any two consecutive Fibonacci numbers are coprime (i.e., their g.c.d. is 1). Prove or disprove: two distinct Fibonacci numbers are coprime if each of them is the product of two distinct primes.

Solution by Lawrence Somer, Washington, D.C.

A counterexample is provided by the Fibonacci numbers

$$F_{22} = 17711 = 89 \cdot 199$$

and

$$F_{121} = 8670007398507948658051921 = 89 \cdot 97415813466381445596089.$$

These numbers were found with the help of Table 1 in [1]. However, the following result is true.

Theorem: Two distinct Fibonacci numbers, each the product of two distinct primes, can have a common factor greater than 1 only if one of the numbers is of the form F_{2p} and the other number is of the form F_{p^2}, where p is an odd prime such that F_p is prime.

Proof: If p is a prime, call p a primitive factor of F_n if $p|F_n$ but $p\nmid F_m$ for $0<m<n$. R. D. Carmichael [2] proved that F_n has a primitive prime factor for every n except $n = 1, 2, 6,$ or 12. In none of these cases is F_n a product of exactly two distinct primes. It is also known that if $m|n$, then $F_m|F_n$. Thus, if n has two or more distinct proper factors r and s which are not equal to 1, 2, 6, or 12, then F_n has at least three prime factors—the primitive prime factors of F_r, F_s, and F_{rs} respectively. Since $F_6 = 8 = 2^3$, it follows that if n is a multiple of 6, then F_n is not a product of two distinct primes. It thus follows that if F_n is a product of two distinct primes, then F_n is of the form F_p, F_{2p}, or F_{p^2}, where p is prime. Moreover, inspection shows that $n \neq 2$ or 4. However, if F_p is a product of two distinct primes, then $(F_n, F_p) > 1$ implies that n is a multiple of p. But then $n = p$ or F_n has at least three distinct prime factors. Further, if p and q are distinct primes, then

$$(F_{2p}, F_{2q}) = (F_{2p}, F_{q^2}) = (F_{p^2}, F_{2q}) = (F_{p^2}, F_{q^2}) = 1.$$

The theorem now follows.

References

1. Brother Alfred Brousseau. *Fibonacci and Related Number Theoretic Tables*. Santa Clara, Calif: The Fibonacci Association, 1972.
2. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $\alpha^n + \beta^n$." *Annals of Mathematics*, 2nd Ser. 15 (1913):30-70.

Also solved by Paul S. Bruckman, Herta T. Freitag, and the proposer.

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