ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be true or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate, signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-345 Proposed by Albert A. Mullin, Huntsville, AL

Prove or disprove: No four consecutive Fibonacci numbers can be products of two distinct primes.

H-346 Proposed by Verner E. Hoggatt, Jr., deceased

Prove or disprove: Let

$$P_1 = 1,\ P_2 = 2,\ P_{n+2} = 2P_{n+1} + P_n \text{ for } n = 1, 2, 3, \ldots,$$

then $P_7 = 169$ is the largest Pell number which is a square, and there are no Pell numbers of the form $2s^2$ for $s > 1$.

H-347 Proposed by Paul S. Bruckman, Sacramento, CA

Prove the identity:

$$\left\{\sum_{n=-\infty}^{\infty} \frac{x^n}{1 + x^{2n}}\right\}^2 = \sum_{n=-\infty}^{\infty} \frac{x^n}{(1 + (-x)^n)^2} \tag{1}$$

valid for all real $x \neq 0, \pm 1$. In particular, prove the identity:

$$\left\{\sum_{n=-\infty}^{\infty} \frac{1}{L_{2n}}\right\}^2 = \sum_{n=-\infty}^{\infty} \frac{1}{L_n^2}. \tag{2}$$

H-348 Proposed by Andreas N. Philippou, Patras, Greece

For each fixed integer $k \geqslant 2$, define the sequence of polynomials $a_n^{(k)}(p)$ by

$$a_n^{(k)}(p) = p^{n+k} \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \left(\frac{1-p}{p}\right)^{n_1 + \cdots + n_k} \quad (n \geqslant 0, -\infty < p < \infty),$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n$. Show that

$$\sum_{n=0}^{\infty} a_n^{(k)}(p) = 1 \quad (0 < p < 1).$$

SOLUTIONS

Are You Curious?

H-327 Proposed by James F. Peters, St. John's University, Collegeville, MN (Vol. 19, No. 2, April 1981)

The sequence

$$1,\ 3,\ 4,\ 6,\ 8,\ 9,\ 11,\ 12,\ 14,\ 16,\ 17,$$
$$19,\ 21,\ 22,\ 24,\ 25,\ 27,\ 29,\ 30,\ 32,\ 34,\ 35,\ \ldots$$

was introduced by D. E. Thoro [Advanced Problem H-12, *The Fibonacci Quarterly* 1, no. 1 (April 1963):54]. Dubbed "A curious sequence," the following is a slightly modified version of the defining relation for this sequence suggested by the Editor [*The Fibonacci Quarterly* 1, no. 1 (Dec. 1963):50]: If

$$T_0 = 1,\ T_1 = 3,\ T_2 = 4,\ T_3 = 6,\ T_4 = 8,\ T_5 = 9,\ T_6 = 11,\ T_7 = 12,$$

then

$$T_{8m+k} = 13m + T_k, \text{ where } k \geqslant 0,\ m = 1, 2, 3, \ldots .$$

Assume

$$F_0 = 1,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}$$

and

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1}$$

and verify the following identities:

$$T_{F_n - 2} = F_{n+1} - 2, \text{ where } n \geqslant 6. \tag{1}$$

For example,

$$T_{F_6 - 2} = T_6 = 11 = F_7 - 2$$
$$T_{F_7 - 2} = T_{11} = 19 = F_8 - 2$$
etc.

$$T_{F_n - 2} - T_{F_{n-2} - 2} = F_n, \text{ where } n \geqslant 6. \tag{2}$$

$$T_{F_n - 2} = F_{n+1} - 2 + L_{n-12}, \text{ where } n \geqslant 15. \tag{3}$$

Solution by Paul S. Bruckman, Concord, CA

We first prove the following explicit formula for T_n:

$$T_n = \left[\frac{13n + 12}{8}\right], \; n = 0, 1, 2, \ldots . \tag{1}$$

Let $U_n = \left[\frac{13n + 12}{8}\right]$. We readily verify that $U_n = T_n$ for $0 \leqslant n \leqslant 7$. Also,

$$U_{8m+k} = \left[\frac{13(8m + k) + 12}{8}\right] = 13m + \left[\frac{13k + 12}{8}\right] = 13m + U_k.$$

Since T_n and U_n satisfy the same recursion and have the same initial values, thereby determining each sequence uniquely, they must coincide. This proves (1).

Next, we will prove the following formula:

$$T_{F_n - 2} = F_{n+1} - 2 + \sum_{k=1}^{m} L_{n-12k}, \; 3 + 12m \leqslant n \leqslant 11 + 12m \tag{2}$$

(if $m = 0$, the sum involving Lucas numbers is considered to vanish). Let

$$G_n = T_{F_n - 2}.$$

Then

$$G_n = \left[\frac{13(F_n - 2) + 12}{8}\right] = \left[\frac{13F_n - 14}{8}\right],$$

or

$$G_n = \left[\frac{13F_n + 2}{8}\right] - 2. \tag{3}$$

Now, using well-known Fibonacci and Lucas identities, it is easy to verify that, for all n,

$$13F_n - 8F_{n+1} = F_7F_n - F_6F_{n+1} = F_{n-6};$$

$$13F_n - 8F_{n+1} - 8L_{n-12} = F_{n-6} - 8L_{n-12} = F_{n-18};$$

$$13F_n - 8F_{n+1} - 8L_{n-12} - 8L_{n-24} = F_{n-18} - 8L_{n-24} = F_{n-30};$$

and, in general,

$$13F_n = 8F_{n+1} + 8\sum_{k=1}^{m} L_{n-12k} + F_{n-6-12m} \text{ (the sum vanishing for } m = 0). \tag{4}$$

Substituting this expression into (3) yields:

$$G_n = F_{n+1} - 2 + \sum_{k=1}^{m} L_{n-12k} + \left[\frac{F_{n-6-12m} + 2}{8}\right], \text{ for all } m, n \geqslant 0. \tag{5}$$

Let $N = n - 6 - 12m$. If $3 + 12m \leqslant n \leqslant 11 + 12m$, then $-3 \leqslant N \leqslant 5$. Hence,

$$-1 = F_{-2} \leqslant F_N \leqslant F_5 = 5 \Rightarrow 1 \leqslant F_N + 2 \leqslant 7 \Rightarrow \left[\frac{F_N + 2}{8}\right] = 0.$$

Thus, for the range $3 + 12m \leqslant n \leqslant 11 + 12m$, the greatest integer term in (5) vanishes, and we are left with (2). It may further be shown that (2) is also valid for $n = 12m + 1$ while, if $n = 12m$ or $12m + 2$, the formula should be reduced by 1 [i.e., the "2" should be replaced by "3" in (2)]. We may therefore obtain an expression which works for *all* values of n:

$$G_n = F_{n+1} - 2 - \chi_n + \sum_{k=1}^{m} L_{n-12k}, \text{ for all } n \geqslant 3 \tag{6}$$

(to avoid negative indices for T_n)

where

$$\chi_n = \begin{cases} 1, & \text{if } n \equiv 0 \text{ or } 2 \pmod{12}; \\ 0, & \text{otherwise}; \end{cases} \quad \text{and } m = [n/12].$$

As a matter of passing interest, we may observe that χ_n may be expressed in terms of familiar functions of n:

$$\chi_n = [n/12] - [(n - 1)/12] + [(n - 2)/12] - [(n - 3)/12]. \tag{7}$$

Furthermore, the sum in (6) may be simplified to the following expression:

$$\sum_{k=1}^{m} L_{n-12k} = \frac{F_{6m}L_{n-6-6m}}{8} \tag{8}$$

The formula in (6) corrects the misstatement of the problem's parts (1) and (3). Thus, part (1) is valid only for $3 \leqslant n \leqslant 11$ and part (3) only for $15 \leqslant n \leqslant 23$ and $n = 13$.

Part (2) of the problem is also false in general. The correct statement of part (2) is as follows:

$$G_n - G_{n-2} = F_n - \theta_n + \sum_{k=1}^{m'} L_{n-1-12k}, \tag{9}$$

where

$$n \geqslant 5; \; \theta_n = \begin{cases} 1, & \text{if } n \equiv 1 \text{ or } 4 \pmod{12}; \\ -1, & \text{if } n \equiv 0 \pmod{12}; \\ 0, & \text{otherwise}; \end{cases} \quad \text{and } m' = [(n - 1)/12].$$

The derivation of (9) is a straightforward consequence of applying (6) and considering the possible residues of (mod 12). Remarks similar to those made after (6) may be made in conjunction with (9). Thus, we see that part (2) of the problem yields the correct formula only for $5 \leqslant n \leqslant 11$.

Also solved by C. Wall and the proposer.

Irrationality

H-328 Proposed by Verner E. Hoggatt, Jr., deceased
(Vol. 19, no. 2, April 1981)

Let θ be a positive irrational number such that $1/\theta + 1/\theta^{j+1} = 1$ ($j \geqslant 1$ and integer). Further, let $A_n = [n\theta]$ and $B_n = [n\theta^{j+1}]$ and $C_n = [n\theta^j]$.

Prove: (a) $A_{C_n} + 1 = B_n$

(b) $A_{C_n+1} - A_{C_n} = 2$

$A_{m+1} - A_m = 1$ ($m \neq C_k$ for any $k > 0$)

(c) $B_n - n$ is the number of A_j's less than B_n.

Solution by Charles R. Wall, Trident Technical College, Charleston, SC

Since $1/\theta + 1/\theta^{j+1} = 1$, $1 = \theta^j(\theta - 1)$ and $1 < \theta < 2$ from elementary considerations.

Now, $n\theta^j - 1 < [n\theta^j] \leqslant n\theta^j$, but the second inequality must be strict, for if $n\theta^j = N$, an integer, then

$$\theta = 1 + 1/\theta^j = 1 + n/N$$

and the left side is irrational but the right side is rational, a contradiction. Thus, $n\theta^j - 1 < [n\theta^j] < n\theta^j$, and multiplying through by $\theta - 1$ yields

$$\begin{aligned} n - 1 < n + 1 - \theta = n\theta^j(\theta - 1) - (\theta - 1) \\ < [n\theta^j](\theta - 1) < n\theta^j(\theta - 1) = n. \end{aligned} \tag{$*$}$$

(a) Note that

$$B_n = [n\theta^{j+1}] = [n(\theta^j + 1)] = [n\theta^j + n] = [n\theta^j] + n.$$

Since $C_n = [n\theta^j]$, we have

$$A_{C_n} = [[n\theta^j]\theta] = [[n\theta^j] + [n\theta^j](\theta - 1)] = [n\theta^j] + n - 1$$

by (*). Therefore, $1 + A_{C_n} = B_n$ as asserted.

(b) Since $A_1 = 1$, the claim that

$$A_{m+1} - A_m = \begin{cases} 2, & \text{if } m = C_k \\ 1, & \text{otherwise} \end{cases}$$

is equivalent to

$$C_k < m \leqslant C_{k+1} \text{ iff } A_m = m + k,$$

a version we shall prove. Now,

$$A_m - m = [m\theta] - m = [m(\theta - 1)] = [m/\theta^j].$$

Let $k = [m/\theta^j]$:

$$m = k\theta^j + r \text{ with } 0 \leqslant r < \theta^j$$

iff
$$m - \theta^j < [m/\theta^j]\theta^j = k\theta^j \leqslant m$$

iff
$$k\theta^j \leqslant m < (k + 1)\theta^j.$$

Taking integral parts, the last inequality is equivalent to

$$C_k = [k\theta^j] < k\theta^j \leqslant m \leqslant [(k + 1)\theta^j] = C_{k+1},$$

which is to say $C_k < m \leqslant C_{k+1}$.

(c) In (a) we noted that $B_n - n = [n\theta^j] = C_n$. From (a), $1 + A_{C_n} = B_n$, so $C_n = B_n - n$ is the number of A's less than B_n.

Also solved by P. Bruckman and the proposer.

<u>*E* Gads</u>

H-329 Proposed by Leonard Carlitz, Duke University, Durham, NC (Vol. 19, No. 2, April 1981)

Show that, for s, t nonnegative integers,

$$(1) \quad e^{-x}\sum_k \frac{x^k}{k!}\binom{k}{s}\binom{k}{t} = \sum_k \frac{x^{s+t-k}}{k!(s-k)!(t-k)!}.$$

More generally, show that

$$(2) \quad e^{-x}\sum_k \frac{x^k}{k!}\binom{k+\alpha}{s}\binom{k}{t} = \sum_k \frac{x^{s+t-k}}{(s-k)!t!}\binom{\alpha+t}{k},$$

and

$$(3) \quad e^{-x}\sum_k \frac{x^k}{k!}\binom{k}{s}\binom{k+\beta}{t} = \sum_k \frac{x^{s+t-k}}{s!(t-k)!}\binom{\beta+s}{k}.$$

Solution by the proposer.

$$e^{-x}\sum_{s,t=0} y^s z^t \sum_k \frac{x}{k!}\binom{k}{s}\binom{k}{t} = e^{-x}\sum_{k=0}^{\infty}\frac{x^k}{k!}(1+y)^k(1+z)^k = e^{xy+xz+xyz}$$
$$= \sum_{k,s,t=0}^{\infty}\frac{(xyz)^k y^s z^t}{k!s!t!}$$
$$= \sum_{s,t=0}^{\infty} y^s z^t \sum_k \frac{x^{s+t-k}}{k!(s-k)!(t-k)!}$$

Equating coefficients of $y^s z^t$, we get (1).

To prove (2), we take

$$e^{-x}\sum_k \frac{x^k}{k!}\binom{k+\alpha}{s}\binom{k}{t} = e^{-x}\sum_k \frac{x^k}{k!}\sum_{i=0}^{s}\binom{\alpha}{i}\binom{k}{s-i}\binom{k}{t}$$
$$= \sum_i \binom{\alpha}{i} e^{-x}\sum_k \frac{x^k}{k!}\binom{k}{s-i}\binom{k}{t}$$

$$= \sum_i \binom{\alpha}{i} \sum_k \frac{x^{s+t-k-i}}{k!(s-k-i)!(t-k)!} \qquad \text{[by (1)]}$$

$$= \sum_k \frac{x^{s+t-k}}{(s-k)!} \sum_i \binom{\alpha}{i} \frac{1}{(k-i)!(t-k+i)!}. \tag{*}$$

The inner sum is equal to

$$\frac{1}{k!(t-k)!} \sum_i \frac{(-k)_i(-\alpha)_i}{i!(t-k+1)_i}$$

$$= \frac{1}{k!(t-k)!} \frac{(\alpha+t-k+1)_k}{(t-k+1)_k} \quad \text{(by Vandermonde's theorem)}$$

$$= \frac{1}{t!}\binom{\alpha+t}{k}.$$

Thus (*) becomes

$$\sum_k \frac{x^{s+t-k}}{(s-k)!t!}\binom{\alpha+t}{k},$$

which proves (2).

The proof of (3) is exactly the same.

REMARK: It does not seem possible to get a simple result for

$$e^{-x} \sum_k \frac{x^k}{k!}\binom{k+\alpha}{s}\binom{k+\beta}{t}.$$

It can be proved that this is equal to the triple sum

$$\sum_{i,j,k} \frac{x^{s+t-k}}{(k-i-j)!(s-k+j)!(t-k+i)!}\binom{\alpha}{i}\binom{\beta}{j}$$

Also solved by P. Bruckman.

O Rats

H-330 Proposed by Verner E. Hoggatt, Jr., deceased
(Vol. 19, No. 4, October 1981)

If θ is a positive irrational number and $1/\theta + 1/\theta^3 = 1$, $A_n = [n\theta]$, $B_n = [n\theta^3]$, $C_n = [n\theta^2]$, then prove or disprove:

$$A_n + B_n + C_n = C_{B_n}.$$

Solution by Paul S. Bruckman, Sacramento, CA

The assertion is false, the first counterexample occurring for $n = 13$. The equation defining θ is equivalent to the cubic: $\theta^3 = \theta^2 + 1$, which has only one real solution:

(1) $\theta = \frac{1}{3}(U + V + 1)$, where $U = \left(\frac{1}{2}(29 + 3\sqrt{93})\right)^{1/3}$, $V = \left(\frac{1}{2}(29 - 3\sqrt{93})\right)^{1/3}$;

thus,

(2) $\theta \doteq 1.4655712,\ \theta^2 \doteq 2.1478989,\ \theta^3 \doteq 3.1478989.$

We find readily that $A_{13} = 19$, $B_{13} = 40$, $C_{13} = 27$, $C_{B_{13}} = C_{40} = 85$; thus

$$A_{13} + B_{13} + C_{13} = 86 \neq 85 = C_{B_{13}}.$$

It is conjectured that the assertion is true for infinitely many n, however. It is further conjectured that $C_{B_n} - (A_n + B_n + C_n) = 0$ or 1 for all n, each occurrence occurring infinitely often, but with "zero" predominating. A proof of this conjecture was not attempted, since it was not required in the solution of the problem; it will probably depend upon the property that $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ partition the natural numbers, and moreover, $B_n = C_n + n$ (both properties readily proved). It is easy to show that

$$|C_{B_n} - (A_n + B_n + C_n)| \leqslant 2 \text{ for all } n,$$

the proof of which depends solely on the properties of the greatest integer function.

Barely There

H-331 Proposed by Andreas N. Philippou, American Univ. of Beirut, Lebanon (Vol. 19, No. 4, October 1981)

For each fixed integer $k \geqslant 2$, define the k-Fibonacci sequence $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ by $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)} & \text{if } 2 \leqslant n \leqslant k \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} & \text{if } n \geqslant k+1. \end{cases}$$

Letting $\alpha = (1 + \sqrt{5})/2$, show:

(a) $f^{(k)} > \alpha^{n-2}$ if $n \geqslant 3$;

(b) $\left\{f^{(k)}\right\}_{n=2}^{\infty}$ has Schnirelmann density 0.

Solution by Paul S. Bruckman, Sacramento, CA

We see that $f_3^{(k)} = 2$ for all $k \geqslant 2$, and $f_n^{(k)} \geqslant F_n + 1$ for all $k \geqslant 3$ and $n \geqslant 4$. Since $2 > \alpha$ and $4 > \alpha$, we see that (a) holds for $n = 3$ and $n = 4$. Also,

$$45 < 49 \Rightarrow 3\sqrt{5} < 7 \Rightarrow 3\sqrt{5} - 5 < 2 \Rightarrow 5^{-1/2} > \frac{1}{2}(3 - \sqrt{5}) = 1 + \beta = \beta^2.$$

Therefore, if $n \geqslant 5$,

$$f_n^{(k)} \geqslant F_n + 1 = 5^{-1/2}(\alpha^n - \beta^n) + 1 > \beta^2(\alpha^n - \beta^n) + 1$$
$$= \alpha^{n-2} + 1 - \beta^{n+2} > \alpha^{n-2}.$$

Hence (a) is true for all $n \geqslant 3$. Q.E.D.

We recall the definition of the Schnirelmann density of a set A of nonnegative integers. If $A(n)$ denotes the number of positive integers in A that are less than or equal to n, then the Schnirelmann density $d(A)$ is given by: $d(A) = \inf\limits_{n \geqslant 1} A(n)/n$.

Let $f^{(k)} = (f_n^{(k)})_{n=0}^{\infty}$ and $A_n^{(k)}$ be the number of positive integers in $f^{(k)}$ that are $\leqslant n$. Since $f_n^{(k)} \geqslant f_n^{\,2}$ for all n and $k \geqslant 2$, it is clear that

$$A_n^{(k)} \leqslant A_n^{(2)};$$

hence $d(f^{(k)}) \leqslant d(f^{(2)})$. It therefore suffices to show that $d(f^{(2)}) = 0$.

Now $A_1^{(2)} = 1$ and $\frac{1}{2}A_2^{(2)} = 1$ (since $F_2 = 1$, $F_3 = 2$, and $f^{(2)}$ is an increasing sequence. Generally, it may be shown that

$$A_n^{(2)} = \left[\frac{\log(1 + n\sqrt{5})}{\log \alpha}\right] - 1.$$

Therefore,

$$d(f^{(2)}) = \inf_{n \geqslant 1}\left\{n^{-1}\left(\left[\frac{\log(1 + n\sqrt{5})}{\log \alpha}\right] - 1\right)\right\} \leqslant \inf_{n \geqslant 1}\left\{n^{-1}\left(\frac{\log(1 + n\sqrt{5})}{\log \alpha} - 1\right)\right\}$$

$$\leqslant \inf_{n \geqslant 1}\left\{n^{-1}\,\frac{\log 2n\alpha}{\log \alpha}\right\} \leqslant \inf_{n \geqslant 3}\left\{\frac{2}{\log \alpha}\cdot\frac{\log n}{n}\right\}.$$

Note that $\log z/z$ is a decreasing function for $z \geqslant 3$ and approaches zero as $z \to \infty$ (z real). Hence,

$$\inf_{z \geqslant 3}(\log z/z) = 0.$$

It follows that $d(f^{(2)}) = 0$. Q.E.D.

Also solved by the proposer.

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