SELF-GENERATING SYSTEMS

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Let $S = a_1, a_2, \ldots,$ and $T = b_1, b_2, \ldots$ be sequences of integers, and let g be an integer. Then gS and $S + T$ denote the sequences $ga_1, ga_2, \ldots$ and $a_1 + b_1, a_2 + b_2, \ldots,$ respectively. Also $\{S\}$ denotes the set $\{a_1, a_2, \ldots\}$.

If the a_n of S are positive and strictly increasing, the characteristic sequence $\chi S = c_1, c_2, \ldots$ has $c_n = 1$ when n is in $\{S\}$ and $c_n = 0$ otherwise. Also ΔS denotes the sequence $d_1, d_2, \ldots$ with $d_n = a_{n+1} - a_n$.

DEFINITION: A system $S_1, S_2, \ldots, S_r$ of sequences of strictly increasing positive integers is *self-generating* if the sets $\{S_1\}, \{S_2\}, \ldots, \{S_r\}$ partition $Z^+ = \{1, 2, 3, \ldots\}$ and there is an $r \times r$ matrix (d_{hk}) with positive integral entries such that

$$\Delta S_h = d_{h1}(\chi S_1) + d_{h2}(\chi S_2) + \cdots + d_{hr}(\chi S_r) \qquad \text{for } 1 \leqslant h \leqslant r.$$

Hoggatt and Hillman in [2] and [3] used shift functions based on certain linear homogeneous recursions to obtain self-generating systems. In Theorem 5 of Section 7 below, we generalize on their work by increasing the set of recursions for which similar results follow. Examples are given in Section 8.

1. THE RECURSIVE SEQUENCE U

In the following, d and $p_1, p_2, \ldots, p_d$ are fixed integers with $d \geqslant 2$ and $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_{d-1} \geqslant p_d = 1$. Also u_n is defined for all integers n by initial conditions

$$u_1 = 1,\ u_0 = u_{-1} = u_{-2} = \cdots = u_{2-d} = 0 \tag{1}$$

and the recursion

$$u_{n+d} = p_1 u_{n+d-1} + p_2 u_{n+d-2} + \cdots + p_d u_n. \tag{2}$$

For each integer i, let U_i denote the sequence $u_{i+1}, u_{i+2}, \ldots$ and let U_0 be written as U.

Hoggatt and Hillman obtained self-generating systems using such recursions for the case $d = 2$ in [3] and for general d with $p_1 = p_2 = \cdots = p_d = 1$ in [2].

In the representations discussed below, we want U to be an increasing sequence of positive integers with 1 as the first term. This is clearly true when $p_1 > 1$. If $p_1 = 1$, then $u_1 = u_2 = 1$ and one of these terms must be deleted; this is equivalent to changing the initial conditions (1) to the conditions $u_h = 2^{h-1}$ for $1 \leqslant h \leqslant d$ of [2]. Since the case $p_1 = 1$ is that of [2], we avoid notational complications by assuming that $p_1 > 1$ in what follows.

The representations introduced next are similar to those of the papers in the special January 1972 issue of this Quarterly as well as those of [2] and [3].

2. CANONICAL REPRESENTATIONS

Let $N = \{0, 1, 2, \ldots\}$. If $X = x_1, x_2, \ldots$ and $Y = y_1, y_2, \ldots$ are sequences of numbers with $x_n = 0$ for $n > h$, let

$$X \cdot Y = x_1y_1 + x_2y_2 + \cdots + x_hy_h.$$

In this section the only properties of $U = u_1, u_2, \ldots$ needed are $u_1 = 1$ and the fact that U is an increasing sequence of integers.

With respect to U, we define inductively for each m in N a sequence $E_m = e_{m1}, e_{m2}, \ldots$ of nonnegative integers as follows. Let all the terms of E_0 be zero. Assume that E_h has been defined for $0 \leqslant h < m$. Since the u_n are unbounded and $u_1 = 1 \leqslant m$, there is a largest k such that $u_k \leqslant m$. For this k, let $t = m - u_k$. Then E_t is defined, and we let $e_{mk} = 1 + e_{tk}$ and $e_{mn} = e_{tn}$ for $n \neq k$. Clearly $E_m \cdot U = m$, i.e., we have the representation

$$m = e_{m1}u_1 + e_{m2}u_2 + \cdots . \tag{3}$$

It is also clear that when $m = u_k$ with $k \geqslant 1$, $e_{mk} = 1$ and $e_{ms} = 0$ for $s \neq k$.

For $n \geqslant 2$, let q_n and r_n be the integers (guaranteed by the division algorithm) such that

$$m - (e_{m,n+1}u_{n+1} + e_{m,n+2}u_{n+2} + \cdots) = q_nu_n + r_n, \quad 0 \leqslant r_n < u_n.$$

Then the definition of E_m implies that

$$q_n = e_{mn} \quad \text{and} \quad r_n = e_{m1}u_1 + e_{m2}u_2 + \cdots + e_{m,n-1}u_{n-1}.$$

Hence

$$e_{m1}u_1 + e_{m2}u_2 + \cdots + e_{m,n-1}u_{n-1} < u_n \text{ for } n \geqslant 2. \tag{4}$$

We next show that (4) and the fact that each e_{mh} is a nonnegative integer characterize E_m.

LEMMA 1: Let $E = e_1, e_2, \ldots$ and $E' = e'_1, e'_2, \ldots$ be sequences of nonnegative integers with $e_n = 0 = e'_n$ for n greater than some r. Also let

$$e_1u_1 + e_2u_2 + \cdots + e_{n-1}u_{n-1} < u_n$$

and

$$e_1'u_1 + e_2'u_2 + \cdots + e_{n-1}'u_{n-1} < u_n \text{ for } n \geqslant 2 \tag{5}$$

and $E \cdot U = E' \cdot U$. *Then* $E = E'$.

PROOF: Since $e_n = 0 = e_n'$ for $n > r$, $E \neq E'$ implies that there is a largest n with $e_n \neq e_n'$, and we let t be this n. Without loss of generality, we let $e_t < e_t'$. Upon deletion of the equal terms in $E \cdot U = E' \cdot U$, we have

$$e_1u_1 + \cdots + e_tu_t = e_1'u_1 + \cdots + e_t'u_t.$$

Since $u_1 = 1$, this implies that $t > 1$. Then

$$\begin{aligned} u_t &\leqslant (e_t' - e_t)u_t = e_t'u_t - e_tu_t \\ &= (e_1u_1 + \cdots + e_{t-1}u_{t-1}) - (e_1'u_1 + \cdots + e_{t-1}'u_{t-1}). \end{aligned}$$

Since each $e_n' \geqslant 0$, this implies that

$$u_t \leqslant e_1u_1 + \cdots + e_{t-1}u_{t-1},$$

contradicting (5) and proving that $E = E'$.

The following definition introduces another characteristic property of the E_m which will be needed below.

DEFINITION: A sequence $E = e_1, e_2, \ldots$ is *compatible* [with respect to the recursion (2)] if, for any h in Z^+ and any integer k with $1 \leqslant k \leqslant d$, the sequence of k differences

$$p_1 - e_{h+k-1},\ p_2 - e_{h+k-2},\ \ldots,\ p_k - e_h \tag{6}$$

has the two following properties:

I. If $h = 1$ or $k = d$, at least one difference in (6) is nonzero.

II. If some difference in (6) is nonzero, the first nonzero difference is positive.

THEOREM 1: For each m *in* Z^+, E_m *is compatible. Also if* $E = e_1, e_2, \ldots$ *is a compatible sequence with* $e_n = 0$ *for* n *greater than some* n_0 *and* $E \cdot U = m$ *then* $E = E_m$.

PROOF: We first show that E_m is compatible. Let $E = E_m$. If $h = 1$ or $k = d$ and all the differences in (6) were zero, then it would follow from (1) and (2) that

$$u_{h+k} = e_{h+k-1}u_{h+k-1} + e_{h+k-2}u_{h+k-2} + \cdots + e_hu_h.$$

Since this would contradict (4), we have shown than I holds.

To prove II, we assume it false and seek a contradiction. Then we can assume that in (6) the first nonzero difference is $p_g - e_{h+k-g}$ and also that $e_{h+k-g} \geqslant 1 + p_g$. These assumptions would imply

$$\sum_{j=h}^{h+k-1} e_j u_j \geqslant \sum_{j=h+k-g}^{h+k-1} e_j u_j \geqslant u_{h+k-g} + \sum_{j=1}^{g} p_j u_{h+k-j}.$$

Here, if one uses the recursion (2) to replace u_{h+k-g} by $\sum_{j=1}^{d} p_j u_{h+k-g-j}$, one finds, since $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_d$, that

$$\sum_{j=h}^{h+k-1} e_j u_j \geqslant \sum_{j=1}^{d} p_j u_{h+k-g-j} + \sum_{j=1}^{g} p_j u_{h+k-j}$$

$$\geqslant \sum_{j=g+1}^{d} p_j u_{h+k-j} + \sum_{j=1}^{g} p_j u_{h+k-j} = u_{h+k}.$$

This contradicts (4), and thus II holds, and E_m is compatible.

Second, assume that E is compatible, the desired n_0 exists, and $E \cdot U = m$. It suffices to show that $u_n > e_1u_1 + e_2u_2 + \cdots + e_{n-1}u_{n-1}$ for $n \geqslant 2$, since this, the hypothesis $E \cdot U = m$, (4), and Lemma 1 imply that $E = E_m$. We prove these inequalities by induction on n. The hypotheses I and II with $h = 1 = k$ imply that $p_1 > e_1$. Hence, $u_2 = p_1 > e_1 = e_1u_1$, and the case $n = 2$ is true. Assume that $n > 2$ and that the desired inequalities are true for 2, 3, ..., $n - 1$. Using I and II, one finds a k in $\{1, 2, \ldots, d\}$ such that

$$p_k \geqslant 1 + e_{n-k} \quad \text{and} \quad p_j = e_{n-j} \quad \text{for } 1 \leqslant j < k. \tag{7}$$

Using the hypothesis of the induction and $n - k < n$, one has

$$u_{n-k} > \sum_{j=1}^{n-k-1} e_j u_j. \tag{8}$$

Using (2), (7), and (8), one sees that

$$u_n = \sum_{j=1}^{d} p_j u_{n-j} \geqslant u_{n-k} + \sum_{j=1}^{k} e_{n-j}u_{n-j} > \sum_{j=1}^{n-k-1} e_j u_j + \sum_{j=1}^{k} e_{n-j}u_{n-j} = \sum_{j=1}^{n-1} e_j u_j.$$

This establishes the desired inequality for n and completes the proof of the theorem.

<u>*LEMMA 2*</u>: *Let* $k \geqslant 1$, $w = u_k$. *Also define the sequence* $F = f_1, f_2, \ldots$ *by*

$$f_1 = p_r - 1, \text{ where } r \in \{1, 2, \ldots, d\} \text{ and } r \equiv k - 1 \pmod{d};$$

$f_n = 0$ *for* $n \geqslant k$;

$f_n = 0$ *for* $n \equiv k \pmod{d}$;

$f_n = p_j$ *when* $k - n \equiv j \pmod{d}$, $1 < n < k$, *and* $n \not\equiv k \pmod{d}$.

Then $E_{w-1} = F$.

PROOF: Obviously F is compatible. Since $p_d = 1$, repeated use of (2) gives

$$u_z = u_{z-qd} + \sum_{h=0}^{q-1} \sum_{k=1}^{d-1} p_k u_{z-hd-k} \text{ for } q \in Z^+. \tag{9}$$

Now let $q \in N$, $r \in \{1, 2, \ldots, d\}$, and $z = qd + r + 1$. Then

$$u_{z-qd} = u_{r+1} = p_1 u_r + p_2 u_{r-1} + \cdots + p_r u_1$$

follows from (2). Hence, (9) can be rewritten as

$$u_z = u_{qd+r+1} = \sum_{h=0}^{q-1} \sum_{k=1}^{d-1} p_k u_{z-hd-k} + \sum_{k=1}^{r} p_k u_{r+1-k}. \tag{10}$$

Now, $F \cdot U = w - 1$ follows from (10), and then Theorem 1 gives us the desired $E_{w-1} = F$.

3. PARTITIONING Z^+

Let $m \in Z^+$. Then $e_{mk} \neq 0$ for some k and we define z_m as follows: if $e_{m1} > 0$, $z_m = 1$, and if $e_{m1} = 0$, then z_m is the largest h such that $e_{ms} = 0$ for $1 \leqslant s < h$. For $1 \leqslant t \leqslant d$, let $V_t = \{m : z_m \equiv t \pmod{d}\}$. Clearly, V_1, V_2, ..., V_d form a partitioning of Z^+.

4. THE SHIFT FUNCTIONS σ^i

Let Z be the set of all integers. Recall that U_i denotes the sequence u_{i+1}, u_{i+2}, ... For each i in Z, let σ^i be the function from N to Z with

$$\sigma^i(m) = E_m \cdot U_i = e_{m1}u_{i+1} + e_{m2}u_{i+2} + \cdots \quad \text{for all } m \text{ in } N.$$

The following properties are easy to verify:

(i) $\sigma^i(m)$ satisfies the recursion (2) for fixed m in N and varying i.

(ii) $\sigma^i(0) = 0$ for all i in Z.

(iii) $\sigma^i(u_k) = u_{k+i}$ for i in Z and k in Z^+.

(iv) $\sigma^{i+1}(m) = \sigma(\sigma^i(m))$ for m and i in N. The proof of this depends on the fact that the canonical representation of $\sigma^i(m)$ is, in fact, E_m shifted i times.

(v) $\sigma^0(m) = m$ for m in N.

5. DIFFERENCING σ^i

For i in Z and m in Z^+, let the backward difference $\nabla\sigma^i(m)$ be defined by

$$\nabla\sigma^i(m) = \sigma^i(m) - \sigma^i(m-1) = E_m \cdot U_i - E_{m-1} \cdot U_i.$$

For i in Z and n in Z^+, let $D_{in} = \nabla\sigma^i(u_n)$. If $u_n = w$, then $E_w = e_1, e_2, \ldots$ with $e_n = 1$ and $e_t = 0$ for $t \neq n$ and $E_{w-1} = f_1, f_2, \ldots, f_{n-1}, 0, 0, \ldots$ with the f_j as described in Lemma 2. Then

$$D_{in} = u_{i+n} - \sum_{j=1}^{n-1} f_j u_{i+j}.$$

Let $n \equiv k \pmod{d}$ with k in $\{1, 2, \ldots, d\}$. Temporarily, let $i \geqslant 2$. Then, using (10) with $z = i + n$, the formulas of Lemma 2 for the f_j, and the recursion (2), one finds that

$$D_{in} = u_{i+1} \quad \text{if } k = 1, \tag{11}$$

and if $k \neq 1$,

$$\begin{aligned} D_{in} &= u_{i+1} + p_k u_i + p_{k+1} u_{i-1} + \cdots + p_d u_{i+k-d} \\ &= u_{i+1} + u_{i+k} - p_1 u_{i+k-1} - \cdots - p_{k-1} u_{i+1}. \end{aligned} \tag{12}$$

For fixed n and varying i, the D_{in} satisfy the same recursion (2) as the u's. Hence, the truth of (11) and (12) for $i \geqslant 2$ implies these formulas for all integers i. In particular, these formulas imply the following lemma.

LEMMA 3: $D_{in} = D_{ik}$ *if* $n \equiv k \pmod{d}$.

Next we show that $\nabla\sigma^i(m)$ depends only on i and the k such that $m \in V_k$.

THEOREM 2: *Let* $m \in V_k$. *Then* $\nabla\sigma^i(m) = D_{ik}$.

PROOF: Let $E_m = e_1, e_2, \ldots$. Since $m \in V_k$, there is a positive integer z such that $z \equiv k \pmod{d}$, $e_z > 0$, and $e_s = 0$ for $1 \leqslant s < z$. Let $w = e_z$ and $E_{w-1} = f_1, f_2, \ldots, f_{z-1}, 0, 0, \ldots$. Using Theorem 1, one finds that

$$E_{m-1} = f_1, f_2, \ldots, f_{z-1}, e_z - 1, e_{z+1}, e_{z+2}, \ldots$$

and hence,

$$\nabla\sigma^{i}(m) = E_m \cdot U_i - E_{m-1} \cdot U_i = D_{iz}.$$

Then Lemma 3 implies that $\nabla\sigma^{i}(m) = D_{ik}$ as desired.

The two following results are not needed for the main theorem (Theorem 5 below) but they generalize on work of [2] and [3].

<u>LEMMA 4</u>: *For $1 \leqslant i < d$, $\nabla\sigma^{-i}(m)$ is 1 for m in V_{i+1} and is 0 otherwise.*

<u>PROOF</u>: Temporarily, let $k \neq 1$. By Theorem 2 and (12), for m in V_k,

$$\begin{aligned}\nabla\sigma^{-i}(m) &= u_{k-i} - p_1u_{k-i-1} - \cdots - p_{k-1}u_{-i+1} + u_{-i+1}\\ &= (u_{k-i} - p_1u_{k-i-1} - \cdots - p_{k-i}u_0) - p_{k-i+1}u_{-1} - \cdots\\ &\qquad - p_{k-1}u_{-i+1} + u_{-i+1}.\end{aligned}$$

For $k = i + 1$, this becomes

$$\nabla\sigma^{-i}(m) = u_1 - p_1u_0 - p_2u_{-1} - \cdots - p_iu_{-i+1} + u_{-i+1} = u_1 = 1,$$

since

$$u_0 = u_{-1} = \cdots = u_{2-d} = 0.$$

For $k \neq i + 1$, i.e., for m not in V_k, $\nabla\sigma^{-i}(m) = 0$, since

$$u_{k-i} = p_1u_{k-i-1} + \cdots + p_{k-i}u_0$$

by (1) and (2). The same results are obtained for $k = 1$ from (11).

<u>THEOREM 3</u>: *Let $|S|$ denote the number of elements in the set S. Then*

(i) $\sigma^{-i}(m) = |V_{i+1} \cap \{1, 2, \ldots, m\}|$ *for* $i = 1, 2, \ldots, d - 1$.

(ii) $m - \sigma^{-1}(m) - \sigma^{-2}(m) - \cdots - \sigma^{-(d-1)}(m) = |V_1 \cap \{1, 2, \ldots, m\}|$.

<u>PROOF</u>: For (i),

$$\begin{aligned}\nabla\sigma^{-i}(1) + \nabla\sigma^{-i}(2) + \cdots + \nabla\sigma^{-i}(m) &= [\sigma^{-i}(1) - \sigma^{-i}(0)] + [\sigma^{-i}(2) - \sigma^{-i}(1)]\\ &\quad + \cdots + [\sigma^{-i}(m) - \sigma^{-i}(m-1)]\\ &= \sigma^{-i}(m) - \sigma^{-i}(0) = \sigma^{-i}(m).\end{aligned}$$

For fixed i, by Lemma 4, $\nabla\sigma^{-i}(1) + \cdots + \nabla\sigma^{-i}(m)$ is the number of integers in $V_{i+1} \cap \{1, 2, \ldots, m\}$. But the telescoping sum shows this to be $\sigma^{-i}(m)$. Part (ii) follows from (i).

6. A PARTITIONING OF N

For $i = 1, 2, \ldots, d$ and $j = 0, 1, \ldots, p_i - 1$, let B_{ij} be the sequence $b_0, b_1, \ldots$ with $b_m = u_{i+1} + j - p_i + \sigma^i(m)$. When the dependence of b_m on i and j has to be indicated, we will write b_m as b_{ijm}.

THEOREM 4: The $p_1 + p_2 + \cdots + p_d$ subsets $\{B_{ij}\}$ partition N.

PROOF: Let $s \in N$. We need to show that there is a unique ordered triple (i, j, m) such that

$$s = u_{i+1} + j - p_i + \sigma^i(m). \tag{13}$$

Let $E_s = e_1, e_2, \ldots$ and for the sought after m, let $E_m = f_1, f_2, \ldots$, i.e., let $e_{sk} = e_k$ and $e_{mk} = f_k$. With this notation and using (1) and (2), one can rewrite (13) as

$$s = p_1u_i + p_2u_{i-1} + \cdots + p_{i-1}u_2 + p_iu_1 + j - p_i + f_1u_{i+1} + f_2u_{i+2} + \cdots .$$

Since $u_1 = 1$, $p_iu_1 + j - p_i = ju_1$ and the equation takes the form

$$s = ju_1 + p_{i-1}u_2 + p_{i-2}u_3 + \cdots + p_1u_i + f_1u_{i+1} + f_2u_{i+2} + \cdots . \tag{14}$$

Using the condition of Theorem 1 that $E_m = f_1, f_2, \ldots$ must be compatible, together with the fact that $j \leqslant p_i - 1$, one sees that the sequence

$$S = j, p_{i-1}, p_{i-2}, \ldots, p_1, f_1, f_2, \ldots$$

must be compatible. Since the right side of (14) is $S \cdot U$, Theorem 1 (with m replaced by s) tells us that (13) is equivalent to $S = E_s$.

If there is no i with $2 \leqslant i \leqslant d$ and

$$(p_1, p_2, \ldots, p_{i-1}) = (e_i, e_{i-1}, \ldots, e_2) \tag{15}$$

then the sequence $e_2, e_3, \ldots$ is compatible and $E_s = S$ holds if and only if $i = 1$, $j = e_1$, and the sequence $e_2, e_3, \ldots$ is the sequence $f_1, f_2, \ldots$.

Now assume that (15) holds for some i in $\{2, 3, \ldots, d\}$ but not for any larger integer in this set. We wish to show that the sequence

$$e_{i+1}, e_{i+2}, \ldots \tag{16}$$

is compatible. Since $e_1, e_2, \ldots$ is compatible, (16) can fail to be compatible only if there is an integer g with

$$(p_1, p_2, \ldots, p_g) = (e_{i+g}, e_{i+g-1}, \ldots, e_{i+1}) \text{ and } i \leqslant g < d. \tag{17}$$

Then condition II (of the definition of a compatible sequence) with $h = i$ and

$k = 1 + g$ would imply that $e_i \leqslant p_{g+1}$. If $e_i < p_{g+1}$, (15) gives us the contradiction $p_1 = e_i < p_{g+1} \leqslant p_1$. Now condition I implies that $g + 1 < d$. Also $e_i = p_{g+1}$ similarly implies that $p_1 = p_2 = \cdots = p_{g+1}$. This, (17), and the equality $p_1 = e_i$ from (15) would give us

$$(p_1, p_2, \ldots, p_{g+1}) = (e_{i+g}, e_{i+g-1}, \ldots, e_i).$$

As before, condition II with $h = i - 1$ and $k = 2 + g$ implies that $p_{g+2} = p_1$, and hence that

$$(p_1, p_2, \ldots, p_{g+2}) = (e_{i+g}, e_{i+g-1}, \ldots, e_{i-1}).$$

This process would continue until we had

$$(p_1, p_2, \ldots, p_{i+g-1}) = (e_{i+g}, e_{i+g-1}, \ldots, e_2),$$

which contradicts the fact that the i in (15) is maximal.

Hence $e_{i+1}, e_{i+2}, \ldots$ satisfies I and II and so is compatible. Then $E_s = S$ holds if and only if i is the maximal i for (15), $j = e_1$, and

$$f_1, f_2, \ldots = e_{i+1}, e_{i+2}, \ldots .$$

This completes the proof.

7. SELF-GENERATING SYSTEM

For $i = 1, 2, \ldots, d$ and $j = 1, 2, \ldots, p_i$, let A_{ij} be the sequence

$$a_{ij1}, a_{ij2}, \ldots$$

with $a_{ijm} = 1 + b_{i,j-1,m-1}$ (the b's are as in Section 6). When both i and j are known from the context, we may write a_{ijm} as a_m.

THEOREM 5: The sequences A_{ij} for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant p_i$ form a self-generating system.

PROOF: From the definition of the sets $\{B_{i,j-1}\}$ in Section 6 and V_k in Section 3, it follows that

$$V_1 = \{A_{d1}\} \cup T, \tag{18}$$

where T is the union of the $\{A_{ij}\}$ for $1 \leqslant i < d$ and $1 \leqslant j < p_i$, and that

$$V_{h+1} = \{A_{h,p_h}\} \text{ for } h = 1, 2, \ldots, d - 1.$$

Since the $\{B_{ij}\}$ form a partition of N (or, equivalently, since the V's partition Z^+), the $\{A_{ij}\}$ partition Z^+. Since $b_{ijm} = u_{i+1} + j - p_i + \sigma^i(m)$,

$$\begin{aligned}\nabla b_{ijm} = b_{i,j,m} - b_{i,j,m-1} &= (u_{i+1} + j - p_i + \sigma^i(m)) - (u_{i+1} + j - p_i + \sigma^i(m-1)) \\ &= \sigma^i(m) - \sigma^i(m-1) = \\ &= \nabla\sigma^i(m).\end{aligned}$$

Then by Theorem 2 we have

$$\nabla b_{ijm} = \nabla\sigma^i(m) = D_{ik} \text{ if } m \in V_k.$$

Since $a_{ijm} = 1 + b_{i,j-1,m-1}$, ΔA_{ij} is the sequence $d_1, d_2, \ldots$ with

$$d_m = a_{i,j,m+1} - a_{i,j,m} = b_{i,j-1,m} - b_{i,j-1,m-1} = D_{ik}$$

when $m \in V_k$. Since each V_k is an $\{A_{ij}\}$ or a union of $\{A_{ij}\}$,

$$\Delta A_{ij} = \sum_{\substack{1 \leq h \leq d \\ 1 \leq k \leq p_h}} d_{ijhk}\chi A_{hk}$$

where $d_{ijhk} = D_{is}$ when $\{A_{hk}\}$ is a subset of V_s.

8. EXAMPLE

For $d = 3$ and $p_1 = p_2 = 3$, $p_3 = 1$, we have $u_{n+3} = 3u_{n+2} + 3u_{n+1} + u_n$ and $U = 1, 3, 12, 46, 177, \ldots$. As an illustration of the canonical representation in Section 1, for $m = 136$, we have $E_m = 2, 2, 3, 2, 0, 0, \ldots$ and $\sigma(m) = 2u_2 + 2u_3 + 3u_4 + 2u_5 = 522$. The following is a table of the $\sigma^i(m)$ for the i's involved in Theorem 5.

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\sigma(m)$	0	3	6	12	15	18	24	27	30	36	39	42	46
$\sigma^2(m)$	0	12	24	46	58	70	92	104	116	138	150	162	177
$\sigma^3(m)$	0	46	92	177	223	269	354	...					

The $p_1 + p_2 + p_3 = 7$ subsets partitioning Z^+ are:

$\{A_{11}\} = \{\sigma(m) + 1\} = \{1, 4, 7, 13, 16, 19, 25, 28, 31, 37, 40, \ldots\}$

$\{A_{12}\} = \{\sigma(m) + 2\} = \{2, 5, 8, 14, 17, 20, 26, 29, 32, 38, 41, \ldots\}$

$\{A_{13}\} = \{\sigma(m) + 3\} = \{3, 6, 9, 15, 18, 21, 27, 30, 33, 39, 42, \ldots\}$

$\{A_{21}\} = \{\sigma^2(m) + 10\} = \{10, 22, 34, 56, 68, 80, 102, \ldots\}$

$\{A_{22}\} = \{\sigma^2(m) + 11\} = \{11, 23, 35, 57, 69, 81, 103, \ldots\}$

$\{A_{23}\} = \{\sigma^2(m) + 12\} = \{12, 24, 36, 58, 70, 82, 104, \ldots\}$

and

$\{A_{31}\} = \{\sigma^3(m) + 46\} = \{46, 92, 138, 223, \ldots\}.$

The following is a table of D_{ik} for $-2 \leqslant i \leqslant 3$ and $1 \leqslant k \leqslant 3$.

$k \backslash i$	-2	-1	0	1	2	3
1	0	0	1	3	12	46
2	0	1	1	6	22	85
3	1	0	1	4	15	58

Since $V_1 = A_{11} \cup A_{12} \cup A_{21} \cup A_{22} \cup A_{31}$, $V_2 = A_{13}$, and $V_3 = A_{23}$, we have

$$\begin{aligned}\Delta A_{1j} &= D_{11}(\chi A_{11}) + D_{11}(\chi A_{12}) + D_{12}(\chi A_{13}) + D_{11}(\chi A_{21}) \\ &\quad + D_{11}(\chi A_{22}) + D_{13}(\chi A_{23}) + D_{11}(\chi A_{31})\end{aligned}$$

$$\begin{aligned}\Delta A_{2j} &= D_{21}(\chi A_{11}) + D_{21}(\chi A_{12}) + D_{22}(\chi A_{13}) + D_{21}(\chi A_{21}) \\ &\quad + D_{21}(\chi A_{22}) + D_{23}(\chi A_{23}) + D_{21}(\chi A_{31})\end{aligned}$$

$$\begin{aligned}\Delta A_{3j} &= D_{31}(\chi A_{11}) + D_{31}(\chi A_{12}) + D_{32}(\chi A_{13}) + D_{31}(\chi A_{21}) \\ &\quad + D_{31}(\chi A_{22}) + D_{33}(\chi A_{23}) + D_{31}(\chi A_{31})\end{aligned}$$

and the 7×7 matrix (d_{hk}) for the self-generating system A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{31} is

$$\begin{pmatrix} 3 & 3 & 6 & 3 & 3 & 4 & 3 \\ 3 & 3 & 6 & 3 & 3 & 4 & 3 \\ 3 & 3 & 6 & 3 & 3 & 4 & 3 \\ 12 & 12 & 22 & 12 & 12 & 15 & 12 \\ 12 & 12 & 22 & 12 & 12 & 15 & 12 \\ 12 & 12 & 22 & 12 & 12 & 15 & 12 \\ 46 & 46 & 85 & 46 & 46 & 58 & 46 \end{pmatrix}$$

As an illustration of Theorem 3(i), with $i = 1$ and $m = 20$,

$$\begin{aligned}\sigma^{-1}(20) &= \sigma^2(20) - 3\sigma(20) - 3\sigma^0(20) \\ &= 2u_3 + 2u_4 + u_5 - 3(2u_2 + 2u_3 + u_4) - 60 \\ &= 5 = |V_2 \cap \{1, 2, \ldots, 20\}|,\end{aligned}$$

where $V_2 = \{n : z_n \equiv 2 \pmod 3\} = \{3, 6, 9, 15, 18\}$ since the only sequences E_n, with $n \leqslant 20$ and $z_n \equiv 2 \pmod 3$ are:

$$E_3 = 0,\ 1,\ 0,\ 0,\ \ldots$$
$$E_6 = 0,\ 2,\ 0,\ 0,\ \ldots$$
$$E_9 = 0,\ 3,\ 0,\ 0,\ \ldots$$
$$E_{15} = 0,\ 1,\ 1,\ 0,\ \ldots$$
$$E_{18} = 0,\ 2,\ 1,\ 0,\ \ldots$$

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4. See the special issue of *The Fibonacci Quarterly* (Vol. 10, No. 1 [1972]) on Representations.

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