THE EXISTENCE OF K ORTHOGONAL LATIN K-CUBES OF ORDER 6

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INTRODUCTION

A Latin cube of order n is an n^3 ($n \times n \times n$) array in which each of the numbers 1, 2, ..., n appears exactly once in each line of the array. Similarly, a Latin k-cube of order n is an n^k array where each of the numbers 1, 2, ..., n appears exactly once in each line. A set of k Latin k-cubes is orthogonal if, when superimposed, each ordered k-tuple of the numbers 1, 2, ..., n appears once.

Orthogonal Latin k-cubes of order n can be constructed from 2 orthogonal Latin squares of order λ [1]. However, there are no orthogonal Latin squares of order 6 [3] and it has been conjectured that there are thus no orthogonal Latin k-cubes or order 6 [4].

We now show how orthogonal Latin k-cubes can be constructed from three orthogonal Latin cubes.

THEOREM: *If there exist three orthogonal Latin cubes and k orthogonal Latin k-cubes or order n, then there exist orthogonal Latin $(k + 2)$-cubes of order n.*

PROOF: Let $A = (a_{ijk})$, $B = (b_{ijk})$, and $C = (c_{ijk})$ be orthogonal Latin cubes and $A^1, A^2, \ldots, A^k$ be orthogonal Latin k-cubes of order n. Write the entries of A^j as $a^j_{i_1, \ldots, i_k}$.

Then we can define $(k+2)$ orthogonal Latin $(k+2)$-cubes $B^1, B^2, \ldots, B^{k+2}$ by

$$b^1_{i_1,\ldots,i_{k+2}} = a_{a^1_{i_1,\ldots,i_k}, i_{k+1}, i_{k+2}}$$

$$\vdots$$

$$b^k_{i_1,\ldots,i_{k+2}} = a_{a^1_{i_1,\ldots,i_k}, i_{k+1}, i_{k+2}}$$

$$b^{k+1}_{i_1,\ldots,i_{k+2}} = b_{a^1_{i_1,\ldots,i_k}, i_{k+1}, i_{k+2}}$$

$$b^{k+2}_{i_1,\ldots,i_{k+2}} = c_{a^1_{i_1,\ldots,i_k}, i_{k+1}, i_{k+2}}$$

The $(k + 2)$-cubes thus defined are orthogonal, since there is a unique position $(i_1, i_2, \ldots, i_{k+2})$ in each Latin $(k + 2)$-cube for every $(k + 2)$-tuple of the numbers $(1, 2, \ldots, n)$ (see [1]).

Examples of 3 orthogonal Latin 3-cubes and 4 orthogonal Latin 4-cubes of order 6 are presented in Table 1 below. Hence, we have shown the existence of k orthogonal Latin k-cubes of order 6.

TABLE 1

(a) 3 Orthogonal Latin Cubes of Order 6

661	433	526	242	354	115
435	522	663	356	111	244
524	665	431	113	246	352
212	344	155	421	563	636
346	151	214	565	632	423
153	216	342	634	425	561

Other layers are obtained by the cyclic permutation (1 2 3 4 5 6).

(b) 4 Orthogonal Latin 4-Cubes of Order 6

I						II					
3554	2241	1115	5663	4332	6426	1131	3516	2263	6442	5625	4354
1135	3514	2261	6446	5623	4352	2223	1151	3536	4314	6462	5645
2221	1155	3534	4312	6466	5643	3556	2243	1111	5665	4334	6422
4413	5662	6346	3524	1251	2135	6362	4435	5624	2151	3546	1213
6366	4433	5622	2155	3544	1211	5644	6322	4455	1233	2111	3566
5642	6326	4453	1231	2115	3564	4415	5664	6342	3526	1253	2131

III						IV					
2225	1153	3532	4316	6464	5641	6443	5322	4656	2534	3211	1165
3552	2245	1113	5661	4336	6424	4616	6463	5342	1125	2554	3231
1133	3512	2265	6444	5621	4356	5362	4636	6423	3251	1145	2514
5646	6324	4451	1235	2113	3562	1254	3141	2515	6363	5432	4626
4411	5666	6344	3522	1255	2133	2535	1214	3161	4646	6323	5452
6364	4431	5626	2153	3542	1215	3121	2555	1234	5412	4666	6343

V						VI					
4612	6465	5344	1121	2556	3233	5366	4634	6421	3255	1143	2512
5364	4632	6425	3253	1141	2516	6441	5326	4654	2532	3215	1163
6445	5324	4652	2536	3213	1161	4614	6461	5346	1123	2552	3235
2531	1216	3163	4642	6325	5454	3125	2553	1232	5416	4664	6341
3123	2551	1236	5414	4662	6345	1252	3145	2513	6361	5436	4624
1256	3143	2511	6365	5434	4622	2533	1212	3165	4644	6321	5456

Other layers are obtained by the cyclic permutation (1 2 3 4 5 6).

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