ON A CONVOLUTION PRODUCT FOR THE TRANSFORM WHICH MAPS DERIVATIVES INTO DIFFERENCES

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INTRODUCTION

In [1] we defined a linear transform with the property that derivatives are mapped into differences in the following way:

$$V\{f(x)\} = (v_n) = \left(\frac{d^n}{dx^n} e^x f(x)\bigg|_{x=0}\right), \text{ i.e., } v_n = \sum_{i=0}^{n}\binom{n}{i} f^{(i)}(0). \tag{1}$$

Its inverse E transform considered in [2] is defined by:

$$E(e_n) = f(x) = \sum_{i=0}^{+\infty}\frac{\Delta^i e_0}{i!}x^i, \text{ i.e., } f(x) = e^{-x}\sum_{i=0}^{+\infty}\frac{e_i}{i!}x^i, \tag{2}$$

where $\Delta e_n = e_{n+1} - e_n$, $\Delta^k e_n = \Delta(\Delta^{k-1}e_n)$ $(k = 0, 1, \ldots)$.

The linear two-dimensional R transform and its inverse, the I transform, with the property that the partial derivatives are mapped into partial differences are defined in [3] by:

$$R\{f(x, y)\} = (r_{m,n}) = \left(\frac{\partial^{m+n}}{\partial x^m \partial y^n} e^{x+y} f(x, y)\bigg|_{\substack{x=0\\y=0}}\right), \tag{3}$$

$$I(i_{m,n}) = f(x, y) = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{\Delta_m^i\Delta_n^j i_{0,0}}{i!j!}x^i y^j, \tag{4}$$

where

$$\Delta_m i_{m,n} = i_{m+1,n} - i_{m,n},\ \Delta_m^k i_{m,n} = \Delta_m(\Delta_m^{k-1} i_{m,n}),$$

$$\Delta_n i_{m,n} = i_{m,n+1} - i_{m,n},\ \Delta_n^k i_{m,n} = \Delta_n(\Delta_n^{k-1} i_{m,n})\ (k = 0, 1, \ldots).$$

In this paper, we give an extension of the results obtained in [1], [2], and [3]. Having the transform at hand, we proceed to determine a convolution for E and I transforms. Also, we will apply this product to solve some discrete equations by establishing analogies between these equations and corresponding continuous equations. At the end of this paper, we will show the practical use of the described transform for obtaining some combinatorial identities. We use the notation introduced in [1].

1. A CONVOLUTION PRODUCT FOR E AND I TRANSFORMS

Let $C^\infty(R)$ be the set of real functions having continuous derivatives of all orders. Furthermore, let $S_f \subset C^\infty(R)$ be the set where $f \in S_f$ if and only if there exist constants α, $M > 0$ such that $|f^{(k)}(0)| < \alpha M^k$, for every $k \in N_0$, and let S_v be the set of all real sequences where $(v_n) \in S_v$ if and only if there exist constants β, $N > 0$ and $|\Delta^k v_0| < \beta N^k$ for every $k \in N_0$.

DEFINITION 1: Let (v_n), $(w_n) \in S_v$. The convolution product of sequences (v_n) and (w_n) is given by

$$v_n * w_n = \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{n-i} \binom{n}{i} \binom{i}{j} v_j w_{i-j}. \tag{5}$$

It is easy to see that the convolution product can be defined by

$$v_n * w_n = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} \Delta^j v_0 \Delta^{i-j} w_0. \tag{6}$$

If (u_n), (v_n), $(w_n) \in S_v$, then the following properties of convolution product can readily be established:

(a) $c * v_n = cv_n$ (c constant),

(b) $u_n * v_n = v_n * u_n$,

(c) $u_n * (v_n + w_n) = u_n * v_n + u_n * w_n$,

(d) $\Delta^k u_n * v_n = \sum_{i=0}^{k} \binom{k}{i} \Delta^i u_n * \Delta^{k-i} v_n$.

THEOREM 1: (a) *If $f(x) \in S_f$, then $Vf \in S_v$,*

(b) *If $(e_n) \in S_v$, then $E(e_n) \in S_f$,*

(c) *If (u_n), $(v_n) \in S_v$, then $(u_n * v_n) \in S_v$.*

PROOF: (a) By (1), we conclude that

$$|\Delta^k v_0| = |f^{(k)}(0)| < \alpha M^k,$$

and we have that $Vf \in S_v$.

(b) By (2), we conclude that

$$|f^{(k)}(0)| = |\Delta^k e_0| < \beta N^k,$$

and we have that $E(e_n) \in S_f$.

(c) Since (u_n), $(v_n) \in S_v$, it follows that there exist β_1, β_2, N_1, $N_2 > 0$ such that

$$|\Delta^k u_0| < \beta_1 N_1^k \quad \text{and} \quad |\Delta^k v_0| < \beta_2 N_2^k.$$

Using (6), we conclude that

$$\left|\Delta^k (u_n * v_n)\right|_{n=0}\Bigg| < \beta_1\beta_2(N_1 + N_2)^k,$$

which means that $u_n * v_n$ given by (5) or (6) belongs to S_v.

THEOREM 2: Let (v_n), $(w_n) \in S_v$. The relation

$$E(v_n * w_n) = E(v_n)E(w_n) \tag{7}$$

is satisfied if and only if $v_n * w_n$ is defined by (6).

PROOF: If (7) is satisfied, then we will have

$$\Delta^i (v_n * w_n)\Big|_{n=0} = \sum_{j=0}^{i} \binom{i}{j} \Delta^j v_0 \Delta^{i-j} w_0,$$

and hence follows (6). Conversely, if (6) is satisfied, then (7) will follow by elementary series manipulations.

Let $Vf = (v_n)$ and $Vg = (u_n)$. Then by (7) we easily conclude that

$$V\{f(x)g(x)\} = \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{n-i} \binom{n}{i} \binom{i}{j} u_j v_{i-j},$$

$$\text{i.e., } V\{f(x)g(x)\} = (u_n * v_n). \tag{8}$$

Now we consider an extension of the result obtained for V and E transforms to two-dimensional R and I transforms defined by (3) and (4). Theorems for R and I transforms are proved analogously and we omit the proofs here.

Let $C^\infty(R^2)$ be the set of real functions having continuous partial derivatives of all orders with respect to both variables. Also, let $S_f^2 \subset C^\infty(R^2)$ be the set where $f \in S_f^2$ if and only if there exist constants $\alpha, M, N > 0$ such that

$$\left|\frac{\partial^{i+j}}{\partial x^i \partial y^j} f(0, 0)\right| < \alpha M^i N^j,$$

and S_v^2 be the set of real sequences where $(v_{m,n}) \in S_v^2$ if and only if there exist constants β, P, Q and $|\Delta_m^i \Delta_n^j v_{0,0}| < \beta P^i Q^j$ for every i, $j \in N_0$.

DEFINITION 2: Let $(v_{m,n})$, $(w_{m,n}) \in S_v^2$. The convolution product of the sequences $(v_{m,n})$ and $(w_{m,n})$ is given by

$$v_{m,n} * w_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{p=0}^{i} \sum_{q=0}^{j} (-1)^{m+n-i-j}\binom{m}{i}\binom{n}{j}\binom{i}{p}\binom{j}{q} v_{p,q} w_{i-p,j-q}.$$

It is easy to see that the convolution product can be defined by

$$v_{m,n} * w_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{p=0}^{i} \sum_{q=0}^{j} \binom{m}{i}\binom{n}{j}\binom{i}{p}\binom{j}{q} \Delta_m^p \Delta_n^q v_{0,0} \Delta_m^{i-p} \Delta_n^{j-q} w_{0,0}. \quad (9)$$

THEOREM 3: (a) *If* $f(x, y) \in S_f^2$, *then* $R\{f(x, y)\} \in S_v^2$,

(b) *If* $(i_{m,n}) \in S_v^2$, *then* $I(i_{m,n}) \in S_f^2$,

(c) *If* $(i_{m,n}), (r_{m,n}) \in S_v^2$, *then* $(i_{m,n} * r_{m,n}) \in S_v^2$.

THEOREM 4: *Let* $(i_{m,n}), (r_{m,n}) \in S_v^2$. *The relation*

$$I(i_{m,n} * r_{m,n}) = I(i_{m,n})I(r_{m,n}) \quad (10)$$

is satisfied if and only if $i_{m,n} * r_{m,n}$ *is defined by* (9).

Let $R\{f(x, y)\} = (r_{m,n})$ and $Rf(x, y) = (s_{m,n})$. Then by (10) we easily conclude that

$$R\{f(x, y)g(x, y)\} = (r_{m,n} * s_{m,n}). \quad (11)$$

2. SOME APPLICATIONS

2.1 Difference Equations

In this section, we will give some applications of the V, R and its inverse transform in solving some difference and partial difference equations.

From (8) and (11), using the orthogonality relation of the binomial coefficients, we obtain the following relations:

$$V\left\{x^k \frac{d^p f(x)}{dx^p}\right\} = (n^{(k)} \Delta^p v_{n-k})$$

and

$$R\left\{x^k y^k \frac{\partial^{i+j} f(x, y)}{\partial x^i \partial y^j}\right\} = (m^{(k)} n^{(p)} \Delta_m^i \Delta_n^j r_{m-k,n-p}).$$

These relations show that the V and R transform maps linear differential equations with polynomial coefficients to linear difference equations with polynomial coefficients, too. The above correspondence may provide a useful method for solving difference equations with polynomial coefficients because the resulting differential equation is often easier to solve.

2.1.1. By an application of the V transform we conclude that the difference equation which corresponds to the following differential equation

$$(a_1x^2 + b_1x + c_1)y'' + (a_2x^2 + b_2x + c_2)y' + (a_3x^2 + b_3x + c_3)y = 0 \quad (12)$$

is given by

$$\begin{aligned} c_1v_{n+2} + (b_1n - 2c_1 + c_2)v_{n+1} &+ (a_1n(n-1) + (b_2 - 2b_1)n + c_1 - c_2 + c_3)v_n \\ &+ n((a_2 - 2a_1)(n-1) + b_1 - b_2 + b_3)v_{n-1} \\ &+ n(n-1)(a_1 - a_2 + a_3)v_{n-2} = 0. \end{aligned} \quad (13)$$

Equation (13) is a second-order difference equation in one of the following three cases:

1. $b_1 = 0,\ c_1 = c_2 = 0;$
2. $a_1 = a_3,\ a_2 = 2a_1,\ b_1 + b_3 = b_2;$
3. $c_1 = 0,\ a_1 + a_3 = a_2.$

Notice that Equation (12) contains some differential equations of special functions as Legendre's, Laguerre's, Chebyshev's, Hermite's, etc. For example, by an application of V and E transforms to Laguerre and Bessel differential equations and their solutions, we find that the solutions of difference equations

$$(n+1)v_{n+1} + (m - 3n - 1)v_n + 2nv_{n-1} = 0$$

and

$$(n^2 - m^2)v_n - n(2n-1)v_{n-1} + n(n-1)v_{n-2} = 0$$

are given by

$$v_n = m!\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{n}{k},$$

and

$$v_n = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{\pi}\int_0^{\pi}\cos\left(mt + \frac{k\pi}{2}\right)\sin^k t\,dt.$$

<u>2.1.2.</u> By an application of the V transform to the equation

$$(e^x + 1)y' + e^x y = 2ae^{ax} \quad (a \in R) \quad (14)$$

we get the equation

$$v_{n+1} - v_n + \sum_{i=0}^{n}\binom{n}{i}v_{i+1} = 2a(1+a)^n. \quad (15)$$

Since a particular solution of (14), given by

$$y = \frac{2e^{ax}}{e^x + 1}$$

belongs to S_f, we have that a particular solution of (15) is given by

$$v_n = E_n(a + 1),$$

where $E_n(a + 1)$ are Euler's polynomials.

2.1.3. By an application of the V transform to the equation

$$y'' - y' = 2 \sin x,\ y(0) = 2,\ y'(0) = 0 \tag{16}$$

we get the equation

$$v_{n+2} - 3v_{n+1} + 2v_n = 2^{(n/2)+1} \sin\left(n \frac{\pi}{4}\right),\ v_0 = 2,\ v_1 = 2. \tag{17}$$

Since the solution of (16), given by

$$y = e^x + \cos x - \sin x$$

belongs to S_f, we have that the solution of (17) is given by

$$v_n = 2^n + 2^{n/2} \cos\left(n \frac{\pi}{4}\right) - 2^{n/2} \sin\left(n \frac{\pi}{4}\right).$$

2.1.4. The transforms V_1 and E_1, defined by

$$V_1\{f(x)\} = (v_n) = \left(\frac{d^n}{dx^n} e^{x-x_0} f(x)\bigg|_{x=x_0}\right)$$

and

$$E_1(e_n) = f(x) = \sum_{k=0}^{+\infty} \frac{\Delta^k v_0}{k!}(x - x_0)^k$$

have analogous properties to the V and E transforms.

By an application of the E_1 transform to

$$\Delta^m v_n + a_1 \Delta^{m-1} v_n + \cdots + a_m v_n = e_n \quad (a_i \in R,\ i = 1, 2, \ldots, m) \tag{18}$$

we get the equation

$$y^{(m)}(x) + a_1 y^{(m-1)}(x) + \cdots + a_m y(x) = f(x), \tag{19}$$

where $f(x) = E(e_n)$.

In paper [4] (see also [5]), Cauchy obtained that the general solution of Equation (19) is given by

$$y = \sum \operatorname{Res}\left(\frac{f(z)}{g(z)} e^{zx}\right) + \sum \operatorname{Res}\left(\frac{e^{zx}}{g(z)} \int_{x_0}^{x} e^{-zt} f(t)\,dt\right),$$

where $f(z)$ is an arbitrary regular function whose zeros do not coincide with zeros of the polynomial $g(z) = z^m + a_1 z^{m-1} + \cdots + a_m$. The summation is taken over all the singularities of the function

$$\frac{f(z)}{g(z)}\, e^{zx},$$

i.e., over all the zeros of the polynomial $g(z)$.

Since $y(x) \in S_f$, then by an application of the V_1 transform and using the convolution product, i.e., using (8), we have that the solution of linear difference equations (18) is given by

$$v_n = \sum \operatorname{Res} \frac{f(z)}{g(z)}(1 + z)^n + \sum \operatorname{Res}\left((1 + z)^{n-1} \sum_{k=0}^{n-1} (1 + z)^{-k} f_k\right).$$

Notice that B. Tortolini [6] (see also [5]) obtained this result in another way.

2.1.5. By an application of the V transform to the following recurrence relations for Laguerre and Gegenbauer polynomials

$$(m + 1)L_{m+1}^{(\alpha)}(x) - (x - 2m - \alpha - 1)L_m^{(\alpha)}(x) + (m + \alpha)L_{m-1}^{(\alpha)}(x) = 0$$

and

$$(m + 1)G_{m+1}^{(\alpha)}(x) - 2(m + \alpha)xG_m^{(\alpha)}(x) + (m + 2\alpha - 1)G_{m-1}^{(\alpha)}(x) = 0$$

we get that particular solutions of equations

$$(m + 1)v_{m+1,n} - (2m + \alpha - 1)v_{m,n} + (m + \alpha)v_{m-1,n} + nv_{m,n-1} = 0$$

and

$$(m + 1)v_{m+1,n} + (m + 2\alpha - 1)v_{m-1,n} - 2(m + \alpha)nv_{m,n-1} = 0$$

are given, respectively, by

$$v_{m,n} = \sum_{i=0}^{\min(m,n)} (-1)^i \binom{m + \alpha}{m - i}\binom{n}{i} \qquad (\alpha > -1)$$

and

$$v_{m,n} = \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{[m/2]} (-1)^i \frac{2^{m-2i}\,\Gamma(m + \alpha - i)}{i!}\binom{n}{m - 2i} \quad \left(\alpha > -\frac{1}{2},\ \alpha \neq 0\right).$$

2.1.6. By an application of the I transform to the equation

$$Ar_{m+1,n} + Br_{m,n+1} + (C - A - B)r_{m,n} = 0 \qquad (A, B, C \in R) \tag{20}$$

we get the equation

$$Af_x + Bf_y + Cf = 0. \tag{21}$$

Since the general solution of Equation (21), given by

$$f = e^{-(C/A)x} f(Bx - Ay),\ A \neq 0;\ f = e^{-(C/B)y} f(x),\ A = 0,\ B \neq 0,$$

where f is an arbitrary function, belongs to S_f^2, we have, by application of the R transform and the convolution product, i.e., using (11), that the general solution of (20) is given by

$$r_{m,n} = \left(1 - \frac{C}{A}\right)^m \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^j A^j \binom{m}{i}\binom{n}{j}\left(\frac{AB}{A - C}\right)^i a_{i+j} \qquad (A \neq 0,\ A \neq C)$$

$$r_{m,n} = B^m \sum_{i=0}^{n} (-1)^i A^i \binom{n}{i} a_{m+i} \qquad (A \neq 0,\ A = C)$$

$$r_{m,n} = \left(1 - \frac{C}{B}\right)^n a_m \qquad (A = 0,\ B \neq 0)$$

where in all cases a_m is an arbitrary sequence. Compare this with the solutions given by Kečkić [7].

2.1.7. By an application of the I transform to the equation

$$r_{m+1,\,n+1} - 3r_{m+1,\,n} - 4r_{m,\,n+1} + 12r_{m,\,n} = 2^{m+n+1}, \tag{22}$$

we get the equation

$$f_{xy} - 2f_x - 3f_y + 6f = 2e^{x+y}. \tag{23}$$

Since the general solution of Equation (23), given by

$$f(x, y) = (a(x) + b(y))e^{3x+2y} + e^{x+y},$$

where $a(x)$ and $b(y)$ are arbitrary functions, belongs to S_f^2, we have, by an application of the R transform, that the general solution of (22) is given by

$$r_{m,n} = (a_m + b_n) * 4^m 3^n + 2^{m+n},$$

where a_m and b_n are arbitrary sequences.

2.2 Combinatorial Identities

Now we will show that the described transform is very useful for obtaining some combinatorial identities.

Applying the V transform to both sides of relations

$$\sum_{i=0}^{k} L_i^{\alpha}(x) = L_k^{\alpha+1}(x)$$

and

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} L_i(x) = \frac{x^k}{k!},$$

where $L_i^{\alpha}(x)$ are Laguerre polynomials defined by

$$L_i^{\alpha}(x) = \sum_{j=0}^{i} (-1)^j \binom{i+\alpha}{i-j} \frac{x^j}{j!},$$

we can easily obtain the following combinatorial identities:

$$\sum_{i=0}^{k} \sum_{j=0}^{\min(n,i)} (-1)^j \binom{i+\alpha}{i-j}\binom{n}{j} = \sum_{j=0}^{\min(n,k)} (-1)^j \binom{n}{j}\binom{k+\alpha+1}{k-j};$$

$$\sum_{i=0}^{k} \sum_{j=0}^{\min(n,i)} (-1)^{i+j} \binom{k}{i}\binom{i}{j}\binom{n}{j} = \binom{n}{k}.$$

Similarly, by application of the R transform to the relations

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (x+y)^{k-i} y^i = x^k$$

and

$$\sum_{i=0}^{k} \binom{k}{i} (2y)^i H_{k-i}(x) = H_k(x+y),$$

where $H_i(x)$ are Hermite polynomials defined by

$$H_i(x) = \sum_{j=0}^{[i/2]} \frac{(-1)^j i!}{j!(i-2j)!}(2x)^{i-2j},$$

we have the following combinatorial identities:

$$\sum_{i=0}^{\min(n,k)} (-1)^i \binom{n}{i}\binom{m+n-1}{k-i} = \binom{m}{n};$$

$$\sum_{i=0}^{\min(n,k)} \sum_{j=0}^{\left[\frac{k-i}{2}\right]} \frac{(-1)^j}{4^j j!}\binom{n}{i}\binom{m}{k-i-2j} = \sum_{j=0}^{[k/2]} \frac{(-1)^j}{4^j j!}\binom{m+n}{k-2j}$$

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LETTER TO THE EDITOR

A NOTE ON THE GEOMETRY OF THE GREAT PYRAMID

The information in James M. Suttenfield, Jr., "A New Series," *The Fibonacci Quarterly* 16, no. 4 (August 1978):335-343, may be misleading to those who have never studied the geometry of the Great Pyramid.

Mr. Suttenfield apparently used information in recent literature to suggest geometry for the Great Pyramid which is different from well-known theories. Mr. Suttenfield's dimensions yield an angle between a face plane and the base plane:

$$\beta = \arctan \frac{\pi}{2\sqrt{\phi}} = 50°59'58.9'' \quad (\phi = \text{golden number})$$

An error analysis using eight sets of angle data from W. M. F. Petrie, *The Pyramids and Temples of Gizeh* (Longon: Field & Tauer, 1883), yields an average of his mean angles of 51°50'03.25". Considering his uncertainties, the standard deviation (1σ) about the mean is ±02'59.155". A more narrow window of ±01'29.375" can be found by taking the averages of his minimum and maximum angles due to the uncertainties.

The theory that the perimeter of the pyramid divided by twice its vertical height is the value of π gives an angle of 51°51'14.3" which is just inside the upper limit of the more narrow range of uncertainty. The theory that the slant height divided by one-half the basewidth gives the golden number yields an angle of 51°49'38.25", and this is just short of the average mean angle from Petrie's data. Mr. Suttenfield's theory yields an angle that is short of the mean by 50'04.35", and this is far outside the range of uncertainties in the survey data.

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