# ADVANCED PROBLEMS AND SOLUTIONS

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*Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.*

## PROBLEMS PROPOSED IN THIS ISSUE

H-368 *Proposed by Andreas N. Philippou, University of Patras, Patras, Greece*

For any fixed integer $k \geqslant 2$,

$$\sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k + 1}{n_1, \ \ldots, \ n_k, \ 1} = \sum_{\ell=0}^{n} f_{\ell+1}^{(k)} f_{n+1-\ell}^{(k)}, \ n \geqslant 0, \tag{A}$$

where $f_n^{(k)}$ are the Fibonacci numbers of order $k$ [1], [2].

In particular, for $k = 2$,

$$\sum_{\ell=0}^{[n/2]} (n + 1 - \ell)\binom{n - \ell}{\ell} = \sum_{\ell=0}^{n} F_{\ell+1}F_{n+1-\ell}, \ n \geqslant 0. \tag{A.1}$$

The problem also includes as a special case ($k = 1$) the following:

$$\binom{n + r - 1}{r - 1} = \sum_{\ell=0}^{n} \binom{n - \ell + r - 2}{r - 2}, \ n \geqslant 0. \tag{B}$$

References

1. A. N. Philippou & A. A. Muwafi. "Waiting for the $k^{\text{th}}$ Consecutive Success and the Fibonacci Sequence of Order $k$." *The Fibonacci Quarterly* 20, no. 1 (1983):28-32.
2. A. N. Philippou. "A Note on the Fibonacci Sequence of Order $k$ and Multinomial Coefficients." *The Fibonacci Quarterly* 21. no. 2 (1983):82-86.

H-369 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Call an integer-valued arithmetic function $f$ a *gcd sequence* if

$$\gcd(a, b) = d \quad \text{implies} \quad \gcd(f(a), f(b)) = f(d)$$

for all positive integers $a$ and $b$. A gcd sequence is *primitive* if it is neither an integer multiple nor a positive integer power of some other gcd sequence. Examples of primitive gcd sequences include:

(1) $f(n) = 1$ (2) $f(n) = n$ (3) $f(n)$ = largest squarefree divisor of $n$
(4) $f(n) = 2^n - 1$ (5) $f(n) = F_n$ (Fibonacci sequence)

Prove that there are infinitely many primitive gcd sequences.

H-370 *Proposed by M. Wachtel & H. Schmutz, Zurich, Switzerland*

For every positive integer $a$ show that

$$5 \cdot [5 \cdot (a^2 + a) + 1] + 1 \tag{A}$$
$$5 \cdot [5 \cdot [5 \cdot [5(a^2 + a) + 1] + 1] + 1] + 1 \tag{B}$$

are *products of two consecutive integers*, and that no integral divisor of

$$5(a^2 + a) + 1$$

is congruent to 3 or 7, modulo 10.

H-371 *Proposed by Paul S. Bruckman, Carmichael, CA*

Let $[\overline{k}]$ represent the purely periodic continued fraction:

$$k + 1/(k + 1/(k + \ldots,\ k = 1, 2, 3, \ldots .$$

Show that

$$[\overline{k}]^3 = [\overline{k^3 + 3k}]. \tag{1}$$

Generalize to other powers.

## SOLUTIONS

### Give Poly Sum!

H-348 *Proposed by Andreas N. Philippou, Patras, Greece*
*(Vol. 20, no. 4, November 1982)*

For each fixed integer $k \geqslant 2$, define the sequence of polynomials $a_n^{(k)}(p)$ by

$$a_n^{(k)}(p) = p^{n+k} \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \left(\frac{1-p}{p}\right)^{n_1 + \cdots + n_k} \quad (n \geqslant 0,\ -\infty < p < \infty),$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$ such that

$$n_1 + 2n_2 + \cdots + kn_k = n.$$

Show that

$$\sum_{n=0}^{\infty} a_n^{(k)}(p) = 1 \quad (0 < p < 1).$$

*Solution by the proposer.*

Using the definition of $\alpha_n^{(k)}(p)$ and the transformation $n_i = m_i$ $(1 \leqslant i \leqslant k)$ and

$$n = m + \sum_{i=1}^{k} (i - 1)m_i,$$

we get

$$\sum_{n=0}^{\infty} \alpha_n^{(k)}(p) = p^k \sum_{m=0}^{\infty} \sum_{\substack{m_1, \ldots, m_k \ni \\ m_1 + \cdots + m_k = m}} \binom{m_1 + \cdots + m_k}{m_1, \ldots, m_k} p^{m_1 + 2m_2 + \cdots + km_k} \left(\frac{1-p}{p}\right)^{m_1 + \cdots + m_k}$$

$$= p^k \sum_{m=0}^{\infty} \left(\left(\frac{1-p}{p}\right)(p + p^2 + \cdots + p^k)\right)^m, \text{ by the multinomial theorem,}$$

$$= p^k \sum_{m=0}^{\infty} (1 - p^k)^m = 1, \text{ for } |1 - p^k| < 1, \tag{1}$$

which establishes the result. Moreover, (1) implies that for any fixed integer $\ell \geqslant 1$,

$$\sum_{n=0}^{\infty} \alpha_n^{(2\ell)}(p) = 1, \text{ for } -1 < p < 1. \tag{2}$$

Remark: If $p = 1/2$, the problem reduces to showing that

$$\sum_{n=1}^{\infty} (f_n^{(k)}/2^n) = 2^{k-1}, \tag{3}$$

where $f_n^{(k)}$ is the Fibonacci sequence of order $k$, since it may be seen, [1]-[2], that

$$f_{n+1}^{(k)} = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \quad (n \geqslant 0).$$

A direct proof of (3) is given in [3].

References

1. A. N. Philippou & A. A. Muwafi. "Waiting for the $k^{\text{th}}$ Consecutive Success and the Fibonacci Sequence of Order $k$." *The Fibonacci Quarterly* 20, no. 1 (1982):28-32.
2. A. N. Philippou. "A Note on the Fibonacci Sequence of Order $k$ and Multinomial Coefficients." *The Fibonacci Quarterly* 21, no. 2 (1983):82-86.
3. A. N. Philippou. Solution of Problem H-322. *The Fibonacci Quarterly* 20, no. 2 (1982):189-90.

*Also solved by Paul S. Bruckman and L. Kuipers.*

## Triggy

H-349 *Proposed by Paul S. Bruckman, Carmichael, CA*
*(Vol. 21, no. 1, February 1983)*

Define $S_n$ as follows: $S_n \equiv \sum_{k=1}^{n-1} \csc^2 \pi k/n$, $n = 2, 3, \ldots$ . Prove $S_n = \dfrac{n^2 - 1}{3}$.

*Solution by Ömer Eğecioğlu, University of California, La Jolla, CA*

We will prove a slight generalization: Let $\xi$ be a primitive $n^{\text{th}}$ root of 1. Then

$$\sum_{k=0}^{n-1} \xi^{km} = \begin{cases} n & \text{if } n \mid m \\ 0 & \text{otherwise.} \end{cases}$$

For $|t| < 1$, we have

$$\sum_{k=0}^{n-1} \frac{1}{1 - \xi^k t} = \sum_{k=0}^{n-1} \sum_{m=0}^{\infty} \xi^{km} t^m = \sum_{m=0}^{\infty} t^m \sum_{k=0}^{n-1} \xi^{km} = \frac{n}{1 - t^n}.$$

Thus

$$\sum_{k=1}^{n-1} \frac{1}{1 - \xi^k} = \lim_{t \to 1} \frac{n}{1 - t^n} - \frac{1}{1 - t} = \frac{n - 1}{2}.$$

Using the fact that $|1 - \xi^k|^2 = (1 - \xi^k)(1 - \overline{\xi}^k)$ and partial fractions, similar techniques yield the following, more general formula: For $0 < m \leqslant n$, we have

$$\sum_{k=1}^{n-1} \frac{\xi^{km}}{|1 - \xi^k|^2} = \frac{1}{12}(n^2 + 6m(m - n) - 1). \tag{1}$$

Now, writing

$$\xi^k = \cos\frac{2\pi k}{n} + i \sin\frac{2\pi k}{n},$$

we obtain

$$|1 - \xi^k|^2 = (1 - \xi^k)(1 - \overline{\xi}^k) = 2(1 - Re\,\xi^k) = 4\sin^2\frac{\pi k}{n}.$$

Separating $\xi^{km}$ into its real and imaginary parts, (1) implies

$$\sum_{k=1}^{n-1} \cos\frac{2\pi km}{n}\csc^2\frac{\pi k}{n} = \frac{1}{3}(n^2 + 6m(m - n) - 1) \tag{1a}$$

$$\sum_{k=1}^{n-1} \sin\frac{2\pi km}{n}\csc^2\frac{\pi k}{n} = 0 \tag{1b}$$

whenever $0 < m \leqslant n$.

From (1a), we obtain the special case $S_n = \dfrac{n^2 - 1}{3}$ by taking $m = n$. For $n$ even, with $m = n/2$, (1a) yields

$$\sum_{k=1}^{n-1} (-1)^k \csc^2\frac{\pi k}{n} = -\frac{1}{6}(n^2 + 2).$$

The following identities can also be obtained by arguments similar to the derivation of (1):

$$\sum_{k=1}^{n-1} \frac{\xi^{km}}{1 - \xi^k} = m - \frac{n + 1}{2}; \tag{2}$$

$$\sum_{k=1}^{n-1} \frac{\xi^{km}}{(1 - \xi^k)^2} = -\frac{1}{12}(n^2 + 6n - 6mn + 6m^2 - 12m + 5); \tag{3}$$

$$\sum_{k=1}^{n-1} \frac{1}{|1 - \xi^k|^2 - 1} = 1 - \frac{n}{\sqrt{3}}\cot\frac{n\pi}{6}. \tag{4}$$

These yield further trigonometric identities by separating $\xi$ to its real and imaginary components. For instance, from (4), we obtain

$$\sum_{k=1}^{n-1} \frac{1}{4\sin^2\frac{k\pi}{n} - 1} = 1 - \frac{n}{\sqrt{3}}\cot\frac{n\pi}{6},$$

and for $n$ even, (3) with $m = n/2$ gives

$$\sum_{k=1}^{n-1} (-1)^k \cos\frac{2\pi k}{n}\cot\frac{\pi k}{n} = 0.$$

*Also solved by P. Bruckman, W. Janous, S. Klein, D. P. Laurie, B. Prielipp, T. J. Rivlin, and J. Suck.*

## We Have the System

H-351 *Proposed by Verner E. Hoggatt, Jr. (deceased)*
*(Vol. 21, No. 1, February 1983)*

Solve the following system of equations:

$$\begin{aligned}
U_1 &= 1\\
V_1 &= 1\\
U_2 &= U_1 + V_1 + F_2 = 3\\
V_2 &= U_2 + V_1 = 4\\
&\vdots\\
U_{n+1} &= U_n + V_n + F_{n+1} \qquad (n \geqslant 1)\\
V_{n+1} &= U_{n+1} + V_n \qquad (n \geqslant 1)
\end{aligned}$$

*Solution by C. Georghiou, University of Patras, Patras, Greece*

The generating functions of the sequences

$$\{F_{n+1}\}_{n=0}^{\infty},\ \{F_{2n}\}_{n=0}^{\infty},\ \text{and } \{F_{2n+1}\}_{n=0}^{\infty}$$

are, respectively,

$$(1 - x - x^2)^{-1},\ x(1 - 3x + x^2)^{-1},\ \text{and } (1 - x)(1 - 3x + x^2)^{-1}.$$

Let $u(x)$ and $v(x)$ represent the generating functions of the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$, respectively. From the given system we get (since $U_0 = V_0 = 0$):

$$\frac{1}{x}u(x) = u(x) + v(x) + (1 - x - x^2)^{-1} \quad \text{and} \quad \frac{1}{x}v(x) = \frac{1}{x}u(x) + v(x).$$

Then

$$\begin{aligned}
v(x) &= x/(1 - x - x^2)(1 - 3x + x^2) = \frac{1}{2}\,\frac{2 - x}{1 - 3x - x^2} - \frac{1}{2}\,\frac{2 + x}{1 - x - x^2}\\
&= \frac{1 - x}{1 - 3x + x^2} + \frac{1}{2}\,\frac{x}{1 - 3x + x^2} - \frac{1}{1 - x - x^2} - \frac{1}{2}\,\frac{x}{1 - x - x^2}.
\end{aligned}$$

Therefore

$$V_n = F_{2n+1} + \frac{1}{2}F_{2n} - F_{n+1} - \frac{1}{2}F_n$$

and

$$U_n = V_n - V_{n-1} = \frac{1}{2}F_{2n+2} - \frac{1}{2}F_{n+1}.$$

*Also solved by P. Bruckman, W. Janous, L. Kuipers, J. Suck, M. Wachtel, and the proposer.*

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