ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS and SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 Solano Dr., S.E.,; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-520 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

(a) Suppose that one has a table for multiplication (mod 10) in which a, b, ..., j have been substituted for 0, 1, ..., 9 in some order. How many decodings of the substitution are possible?

(B) Answer the analogous question for a table of multiplication (mod 12).

B-521 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

See the previous problem. Find all moduli $m > 1$ for which the multiplication (mod m) table can be decoded in only one way.

B-522 *Proposed by Ioan Tomescu, University of Bucharest, Romania*

Find the number $A(n)$ of sequences $(a_1, a_2, \ldots, a_k)$ of integers a_i satisfying $1 \leqslant a_i < a_{i+1} \leqslant n$ and $a_{i+1} - a_i \equiv 1 \pmod 2$ for $i = 1, 2, \ldots, k - 1$. [Here k is variable, but of course $1 \leqslant k \leqslant n$. For example, the three allowable sequences for $n = 2$ are (1), (2), and (1, 2).]

B-523 *Proposed by Laszlo Cseh and Imre Merenyi, Cluj, Romania*

Let $p, a_0, a_1, \ldots, a_n$ be integers with p a positive prime such that

$$\gcd(a_0, p) = 1 = \gcd(a_n, p).$$

Prove that in $\{0, 1, \ldots, p - 1\}$ there are as many solutions of the congruence

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \equiv 0 \pmod{p}$$

as there are of the congruence

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a \equiv 0 \pmod{p}.$$

B-524 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_n = F_{2n-1}^2 + F_n F_{n-1}(F_{2n-1} + F_n^2) + 3F_n F_{n+1}(F_{2n-1} + F_n F_{n-1})$. Show that S_n is the square of a Fibonacci number.

B-525 *Proposed by Walter Blumberg, Coral Springs, FL*

Let x, y, and z be positive integers such that $2^x - 1 = y^z$ and $x > 1$. Prove that $z = 1$.

SOLUTIONS

Fibonacci-Lucas Centroid

B-496 *Proposed by Stanley Rabinowitz, Digital Equip. Corp., Merrimack, NH*

Show that the centroid of the triangle whose vertices have coordinates

$$(F_n, L_n),\ (F_{n+1}, L_{n+1}),\ (F_{n+6}, L_{n+6})$$

is (F_{n+4}, L_{n+4}).

Solution, independently, by Walter Blumberg, Coral Springs, FL; Wray G. Brady, Slippery Rock, PA; Paul S. Bruckman, Sacramento, CA; Laszlo Cseh, Cluj, Romania; Leonard Dresel, Reading, England; Herta T. Freitag, Roanoke, VA; L. Kuipers, Switzerland; Stanley Rabinowitz, Merimack, NH; Imre Merenyi, Cluj, Romania; John W. Milsom, Butler, PA; Bob Prielipp, Oshkosh, WI; Sahib Singh, Clarion, PA; Lawrence Somer, Washington, CD; Gregory Wulczyn, Lewisburg, PA.

The coordinates (x, y) of the centroid are given by

$$\begin{aligned} 3x &= F_n + F_{n+1} + F_{n+6} \\ &= F_{n+2} + F_{n+4} + F_{n+5} \\ &= F_{n+2} + F_{n+4} + F_{n+3} + F_{n+4} \\ &= 3F_{n+4}, \end{aligned}$$

and similarly,

$$\begin{aligned} 3y &= L_n + L_{n+1} + L_{n+6} \\ &= 3L_{n+4}. \end{aligned}$$

Hence, the centroid is (F_{n+4}, L_{n+4}).

Area of a Fibonacci-Lucas Triangle

B-497 *Proposed by Stanley Rabinowitz, Digital Equip. Corp., Merrimack, NH*

For d an odd positive integer, find the area of the triangle with vertices (F_n, L_n), (F_{n+d}, L_{n+d}), and (F_{n+2d}, L_{n+2d}).

Solution by Paul S. Bruckman, Sacramento, CA

By means of a well-known determinant formula, the area of the given triangle is given by

$$A = \frac{1}{2}\left\|\begin{matrix} 1 & 1 & 1 \\ F_n & F_{n+d} & F_{n+2d} \\ L_n & L_{n+d} & L_{n+2d} \end{matrix}\right\| \tag{1}$$

(In the above expression, the inner bar is the determinant symbol, the outer bar represents absolute value.) Then

$$A = \frac{1}{2}|(F_{n+2d}L_{n+d} - F_{n+d}L_{n+2d}) - (F_{n+2d}L_n - F_nL_{n+2d}) + (F_{n+d}L_n - F_nL_{n+d})|.$$

Using the relation

$$F_uL_v - F_vL_u = 2(-1)^vF_{u-v}, \tag{2}$$

this becomes:

$$A = (-1)^{n+d}F_d - (-1)^nF_{2d} + (-1)^nF_d = (-1)^dF_d - F_{2d} + F_d,$$

which equals F_{2d} when d is odd (and equals $F_{2d} - 2F_d$ when d is even).

Also solved by Walter Blumberg, Wray G. Brady, Leonard Dresel, Herta T. Freitag, L. Kuipers, Graham Lord, John W. Milsom, Bob Prielipp, Sahib Singh, Gregory Wulczyn, and the proposer.

Fibonacci Recursions Modulo 10

B-498 *Proposed by Herta T. Freitag, Roanoke, VA*

Characterize the positive integers k such that, for all positive integers n, $F_n + F_{n+k} \equiv F_{n+2k}$ (mod 10).

Solution by Leonard Dresel, University of Reading, England

When k is *odd*, we have the identity $F_{m+k} - F_{m-k} = F_mL_k$. Applying this with $m = n + k$, we have $F_m \equiv F_mL_k$ (mod 10), and this will be satisfied whenever $L_k \equiv 1$ (mod 10).

On the other hand, when k is even, we have $F_{m+k} - F_{m-k} = L_mF_k$, and it is not possible to satisfy the given recurrence for even k.

Returning to the case of *odd* k, the condition $L_k \equiv 1$ (mod 10) is equivalent to

$$L_k \equiv 1 \pmod 2 \textit{ and } L_k \equiv 1 \pmod 5.$$

The first condition implies that k is *not* divisible by 3; with the help of the Binet formula for L_k, the second condition reduces to $2^{k-1} \equiv 1$ (mod 5), which gives that $k - 1$ in a multiple of 4. Combining these results, we have

$$k = 12t + 1 \textit{ or } k = 12t + 5 \quad (t = 0, 1, 2, 3, \ldots).$$

Also solved by Paul S. Bruckman, Laszlo Cseh, L. Kuipers, Imre Merenyi, Bob Prielipp, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

Lucas Recursions Modulo 12

B-499 *Proposed by Herta T. Freitag, Roanoke, VA*

Do the Lucas numbers analogue of B-498.

Solution by Leonard Dresel, University of Reading, England

We have the identities $L_{m+k} - L_{m-k} = L_m L_k$ when k is odd, and $L_{m+k} - L_{m-k} = 5F_m F_k$ when k is even. Hence, putting $m = n + k$, the relation $L_n + L_{n+k} \equiv L_{n+2k}$ (mod 10) leads, for *odd* k, to $L_m \equiv L_m L_k$ (mod 10), so that we require $L_k \equiv 1$ (mod 10), leading to the same values of k as in B-498 above.

Also solved by Paul S. Bruckman, Laszlo Cseh, L. Kuipers, Imre Merényi, Bob Prielipp, Lawrence Somer, Gregory Wulczyn, and the proposer.

Two Kinds of Divisibility

B-500 *Proposed by Philip L. Mana, Albuquerque, NM*

Let $A(n)$ and $B(n)$ be polynomials of positive degree with integer coefficients such that $B(k)|A(k)$ for all integers k. Must there exist a nonzero integer h and a polynomial $C(n)$ with integer coefficients such that $hA(n) = B(n)C(n)$?

Solution by the proposer.

Using the division algorithm and multiplying by an integer h so as to make all coefficients into integers, one has

$$hA(n) = Q(n)B(n) + R(n), \qquad (*)$$

where $Q(n)$ and $R(n)$ are polynomials in n with integral coefficients and $R(n)$ is either the zero polynomial or has degree less than $B(n)$. The hypothesis that $B(n)|A(n)$ and (*) imply that $B(n)|R(n)$ for all integers n. If $R(n)$ is not the zero polynomial, $R(n)$ has lower degree than $B(n)$ and so

$$\lim_{n \to \infty} [R(n)/B(n)] = 0;$$

also $R(n)$ is zero for only a finite number of integers n. Thus $0 < R(n)/B(n) < 1$ for some large enough n, contradicting $B(n)|R(n)$. Hence $R(n)$ is the zero polynomial and (*) shows that the answer is "yes."

Also solved by Paul S. Bruckman and L. Kuipers.

Doubling Back on a Sequence

B-501 *Proposed by J. O. Shallit & J. P. Yamron, U.C. Berkeley, CA*

Let α be the mapping that sends a sequence $X = (x_1, x_2, \ldots, x_{2k})$ of length $2k$ to the sequence of length k,

$$\alpha(X) = (x_1 x_{2k}, x_2 x_{2k-1}, \ldots, x_k x_{k+1}).$$

Let $V = (1, 2, 3, \ldots, 2^h)$, $\alpha^2(V) = \alpha(\alpha(V))$, etc. Prove that $\alpha(V), \alpha^2(V), \ldots, \alpha^{h-1}(V)$ are all strictly increasing sequences.

Solution by Leonard Dresel, University of Reading, England

Suppose the nubmers a_1, a_2, a_3, a_4 form a strictly increasing sequence, subject to the condition $a_1 + a_4 = a_2 + a_3 = S$, then

$$(a_4 - a_1)^2 > (a_3 - a_2)^2$$

and

$$(a_4 - a_1)^2 = (a_3 + a_2)^2$$

gives

$$-4a_4a_1 > -4a_3a_2$$

hence,

$$a_1a_4 < a_2a_3.$$

Now any two consecutive terms of $\alpha(V)$ are of the form a_1a_4, a_2a_3, with

$$a_1 + a_4 = a_3 + a_2 = 1 + 2^h,$$

so that it follows that $\alpha(V)$ is a strictly increasing sequence.

Next, consider $\alpha^2(V)$. To avoid a notational forest, we shall apply our method to the specific case where $h = 4$, with $V = (1, 2, 3, \ldots, 16)$. Then, using a dot to denote multiplication, we have

$$\alpha(V) = (1 \cdot 16,\ 2 \cdot 15,\ 3 \cdot 14,\ \ldots,\ 8 \cdot 9)$$

$$\begin{aligned}\alpha^2(V) &= (1 \cdot 16 \cdot 8 \cdot 9,\ 2 \cdot 15 \cdot 7 \cdot 10,\ \ldots,\ 4 \cdot 13 \cdot 5 \cdot 12)\\ &= (1 \cdot 8 \cdot 9 \cdot 16,\ 2 \cdot 7 \cdot 10 \cdot 15,\ \ldots,\ 4 \cdot 5 \cdot 12 \cdot 13)\\ &= (b_1 \cdot c_1,\ b_2 \cdot c_2,\ \ldots,\ b_4 \cdot c_4)\end{aligned}$$

where

$$(b_1, b_2, b_3, b_4) = \alpha(1, 2, 3, \ldots, 8)$$

and

$$(c_1, c_2, c_3, c_4) = \alpha(9, 10, 11, \ldots, 16).$$

By our previous argument, b_i is strictly increasing, and similarly c_i is. Thus $\alpha^2(V) = (b_ic_i)$ is a strictly increasing sequence. Similarly, we can show that $\alpha^3(V) = (d_1e_1f_1g_1,\ d_2e_2f_2g_2)$, where

$$(d_1, d_2) = \alpha(1, 2, 3, 4),\ (e_1, e_2) = \alpha(5, 6, 7, 8),\ \text{etc.},$$

is strictly increasing. The above arguments can be generalized to apply to any value of h.

Also solved by Paul S. Bruckman, L. Kuipers, and the proposers.

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