WEIGHTED ASSOCIATED STIRLING NUMBERS

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1. INTRODUCTION

The Stirling numbers of the first and second kind can be defined by

$$\begin{cases} (-\log(1 - x))^k = k! \sum_{n=k}^{\infty} S_1(n, k)x^n/n! \\ (e^x - 1)^k = k! \sum_{n=k}^{\infty} S(n, k)x^n/n! \end{cases} \tag{1.1}$$

These numbers are well known and have been studied extensively. There are many good references for them, including [4, Ch. 5] and [9, Ch. 4, pp. 32-38].

Not as well known are the *associated* Stirling numbers of the first and second kind, which can be defined by

$$\begin{cases} (-\log(1 - x) - x)^k = k! \sum_{n=2k}^{\infty} d(n, k)x^n/n! \\ (e^x - x - 1)^k = k! \sum_{n=2k}^{\infty} b(n, k)x^n/n! \end{cases} \tag{1.2}$$

We are using the notation of Riordan [9] for these numbers. One reason they are of interest is their relationship to the Stirling numbers:

$$\begin{cases} S_1(n, n - k) = \sum_{j=0}^{k} d(2k - j, k - j)\binom{n}{2k - j} \\ S(n, n - k) = \sum_{j=0}^{k} b(2k - j, k - j)\binom{n}{2k - j} \end{cases} \tag{1.3}$$

Equations (1.3) prove that $S_1(n, n - k)$ and $S(n, n - k)$ are both polynomials in n of degree $2k$. Combinatorially, $d(n, k)$ is the number of permutations of $Z_n = \{1, 2, \ldots, n\}$ having exactly k cycles such that each cycle has at least two elements; $b(n, k)$ is the number of set partitions of Z_n consisting of exactly k blocks such that each block contains at least two elements. Tables for $d(n, k)$ and $b(n, k)$ can be found in [9, pp. 75-76].

Carlitz [1], [2], has generalized $S_1(n, k)$ and $S(n, k)$ by defining *weighted* Stirling numbers $\overline{S}_1(n, k, \lambda)$ and $\overline{S}(n, k, \lambda)$, where λ is a parameter. Carlitz has also investigated the related functions

$$\begin{cases} R_1(n, k, \lambda) = \overline{S}_1(n, k + 1, \lambda) + S_1(n, k) \\ R(n, k, \lambda) = \overline{S}(n, k + 1, \lambda) + S(n, k) \end{cases} \tag{1.4}$$

For all of these numbers, Carlitz has found generating functions, combinatorial interpretations, recurrence formulas, and other properties. See [1] and [2] for details.

The purpose of this paper is to define, in an appropriate way, the weighted associated Stirling numbers $\bar{d}(n, k, \lambda)$ and $\bar{b}(n, k, \lambda)$, and to examine their properties. In particular, we have the following relationships to $\bar{S}_1(n, k, \lambda)$ and $\bar{S}(n, k, \lambda)$:

$$\begin{cases} \bar{S}_1(n, n-k, \lambda) = \sum_{j=0}^{k} \binom{n}{2k-j+1} \bar{d}(2k-j+1, k-j+1, \lambda) \\ \qquad\qquad + n\lambda S_1(n-1, n-1-k) \\ \bar{S}(n, n-k, \lambda) = \sum_{j=0}^{k} \binom{n}{2k-j+1} \bar{b}(2k-j+1, k-j+1, \lambda) \\ \qquad\qquad + n\lambda S(n-1, n-1-k) \end{cases} \tag{1.5}$$

We also define and investigate related functions $Q_1(n, k, \lambda)$ and $Q(n, k, \lambda)$, which are analogous to $R_1(n, k, \lambda)$ and $R(n, k, \lambda)$. In particular, we define $Q_1(n, k, \lambda)$ and $Q(n, k, \lambda)$ so that

$$\begin{cases} R_1(n, n-k, \lambda) = \sum_{j=0}^{k} Q_1(2k-j, k-j, \lambda)\binom{n}{2k-j} \\ R(n, n-k, \lambda) = \sum_{j=0}^{k} Q(2k-j, k-j, \lambda)\binom{n}{2k-j} \end{cases} \tag{1.6}$$

which can be compared to (1.3).

The development of the weighted associated Stirling numbers will parallel as much as possible the analogous work in [1] and [2]. In addition to the relationships mentioned above, we shall find generating functions, combinatorial interpretations, recurrence formulas, and other properties of $\bar{d}(n, k, \lambda)$, $\bar{b}(n, k, \lambda)$, $Q_1(n, k, \lambda)$, and $Q(n, k, \lambda)$.

2. THE FUNCTIONS $\bar{d}(n, k, \lambda)$ AND $\bar{b}(n, k, \lambda)$

Let n, k be positive integers, $n \geqslant k$, and $k_2, k_3, \ldots, k_n$ nonnegative such that

$$\begin{cases} k = k_2 + k_3 + \cdots + k_n \\ n = 2k_2 + 3k_3 + \cdots + nk_n. \end{cases} \tag{2.1}$$

Put

$$b(n; k_2, \ldots, k_n; \lambda) = \sum (k_2\lambda^2 + k_3\lambda^3 + \cdots + k_n\lambda^n) \tag{2.2}$$

where the summation is over all the partitions of $Z_n = \{1, 2, \ldots, n\}$ into k_2 blocks of cardinality 2, k_3 blocks of cardinality 3, ..., k_n blocks of cardinality n. Then, following the method of Carlitz [1], we sum on both sides of (2.2) and obtain, after some manipulation,

$$\sum_{n=1}^{\infty} \frac{x}{n!} \sum_{k_1, k_2, \ldots} b(n; k_2, k_3, \ldots; \lambda)y^k = y(e^{\lambda x} - \lambda x - 1)\exp\{y(e^x - x - 1)\}. \tag{2.3}$$

Now we define

$$b(n, k, \lambda) = \sum\sum(k_2\lambda^2 + k_3\lambda^3 + \cdots + k_n\lambda^n), \tag{2.4}$$

(2.4)

where the inner summation is over all partitions of Z_n into k_2 blocks of cardinality 2, k_3 blocks of cardinality 3, ..., k_n blocks of cardinality n; the outer summation is over all $k_2, k_3, \ldots, k_n$ satisfying (2.1).

By (2.3) and (2.4), we have

$$\sum_{n,k} \overline{b}(n, k, \lambda)\frac{x^n}{n!} y^k = y(e^{\lambda x} - \lambda x - 1)\exp\{y(e^x - x - 1)\}, \tag{2.5}$$

and from (2.5) we obtain

$$k!\sum_{n=0}^{\infty} \overline{b}(n, k+1, \lambda)\frac{x^n}{n!} = (e^{\lambda x} - \lambda x - 1)(e^x - x - 1)^k. \tag{2.6}$$

It follows from (1.2) and (2.6) that

$$\overline{b}(n, k, \lambda) = \sum_{m=2}^{n-2k+2} \binom{n}{m}\lambda^m b(n-m, k-1). \tag{2.7}$$

For $\lambda = 1$, (2.6) reduces to

$$k!\sum_{n=0}^{\infty} \overline{b}(n, k+1, 1)\frac{x^n}{n!} = (e^x - x - 1)^{k+1} = (k+1)!\sum_{n=0}^{\infty} b(n, k+1)\frac{x^n}{n!}.$$

Thus, we have

$$\overline{b}(n, k, 1) = kb(n, k). \tag{2.8}$$

We also have, by (2.6) and (2.7),

$$\begin{aligned}
\overline{b}(n, 0, \lambda) &= 0,\\
\overline{b}(n, 1, \lambda) &= \lambda \quad \text{if } n \geqslant 2,\\
\overline{b}(n, 2, \lambda) &= \binom{n}{2}\lambda^2 + \binom{n}{3}\lambda^3 + \cdots + \binom{n}{n-2}\lambda^{n-2},\\
\overline{b}(n, k, \lambda) &= 0 \text{ if } n < 2k,\\
\overline{b}(2k, k, \lambda) &= \binom{2k}{2}b(2k-2, k-1)\lambda^2.
\end{aligned}$$

The relationship to $\overline{S}(n, k, \lambda)$ is most easily proved by using an extension of a theorem in [7]. In a forthcoming paper [8], we prove the following:

Theorem 2.1: For $r \geqslant 1$ and $f \neq 0$, let

$$F(x) = \sum_{i=r}^{\infty} f_i \frac{x^i}{i!} \quad \text{and} \quad W(x, \lambda) = 1 + \sum_{t=1}^{\infty} w_t(\lambda)\frac{x^t}{t!}$$

be formal power series. Define $B_{n,j}^{(\lambda)}(0, \ldots, 0, f_r, f_{r+1}, \ldots)$ by

$$W(x, \lambda)(F(x))^j = j!\sum_{n=0}^{\infty} B_{n,j}^{(\lambda)}(0, \ldots, 0, f_r, f_{r+1}, \ldots)\frac{x^n}{n!}.$$

Then $\left(\frac{r!}{f_r}\right)^n B_{k+rn,n}^{(\lambda)}(0, \ldots, 0, f_r, f_{r+1}, \ldots) = (k+rn)(k+rn-1) \ldots (n+1)$

$$\cdot \sum_{j=0}^{k} \frac{n(n-1) \ldots (n-j+1)}{(k+rj)!}\left(\frac{r!}{f_r}\right)^j B_{k+rj,j}^{(\lambda)}(0, \ldots, 0, f_{r+1}, \ldots).$$

It follows from Theorem 2.1 and the generating function for $\overline{S}(n, k, \lambda)$ that if we define

$$(e^{\lambda x} - 1)(e^x - x - 1)^k = k!\sum_{n=0}^{\infty} \overline{a}(n, k + 1, \lambda)\frac{x^n}{n!}, \tag{2.9}$$

then

$$\overline{S}(n, n - k, \lambda) = \sum_{j=0}^{k} \binom{n}{2k - j + 1}\overline{a}(2k - j + 1, k - j + 1, \lambda). \tag{2.10}$$

By (2.6) and (2.9),

$$\overline{a}(n, k + 1, \lambda) = \overline{b}(n, k + 1, \lambda) + \lambda n b(n - 1, k),$$

and by (1.3), (2.10) can be wrtiten

$$\overline{S}(n, n - k, \lambda) = \sum_{j=0}^{k} \overline{b}(2k - j + 1, k - j + 1, \lambda)\binom{n}{2k - j + 1} + \lambda n S(n - 1, n - 1 - k), \tag{2.11}$$

which proves $\overline{S}(n, n - k, \lambda)$ is a polynomial in n of degree $2k + 1$.

It is convenient to define

$$Q(n, k, \lambda) = \overline{b}(n, k + 1, \lambda) + n\lambda b(n - 1, k) + b(n, k), \tag{2.12}$$

which implies

$$Q(n, k, \lambda) = \sum_{m=0}^{n-2k} \binom{n}{m} b(n - m, k)\lambda^m. \tag{2.13}$$

Note that $Q(n, k, 0) = b(n, k)$.

A generating function can be found. If we sum on both sides of (2.12), we have

$$\sum_{n,k} Q(n, k, \lambda)\frac{x^n}{n!} y^k = e^{\lambda x}\exp\{y(e^x - x - 1)\}. \tag{2.14}$$

If we differentiate both sides of (2.14) with respect to y and compare the coefficients of $x^n y^k$, we have

$$Q(n, k, \lambda + 1) = Q(n, k, \lambda) + (k + 1)Q(n, k + 1, \lambda) + nQ(n - 1, k, \lambda). \tag{2.15}$$

If we differentiate both sides of (2.14) with respect to x, we have

$$Q(n + 1, k, \lambda) = \lambda Q(n, k, \lambda) + Q(n, k - 1, \lambda + 1) - Q(n, k - 1, \lambda). \tag{2.16}$$

Combining (2.15) and (2.16), we have our main recurrence formula:

$$Q(n + 1, k, \lambda) = (\lambda + k)Q(n, k, \lambda) + nQ(n - 1, k - 1, \lambda). \tag{2.17}$$

It follows from (3.4) that

$$Q(n, k, 1) = b(n, k) + b(n + 1, k).$$

We also have

$$Q(n, 0, \lambda) = \lambda^n,$$

$$Q(n, 1, \lambda) = \binom{n}{0}\lambda^0 + \binom{n}{1}\lambda^1 + \cdots + \binom{n}{n - 2}\lambda^{n-2},$$

$$Q(n, k, 0) = b(n, k),$$

$$Q(2k, k, \lambda) = b(2k, k),$$

$$Q(n, k, \lambda) = 0 \text{ if } n < 2k.$$

A small table of values is given below.

$Q(n, k, \lambda)$

$n \backslash k$	0	1	2	3
0	1			
1	λ			
2	λ^2	1		
3	λ^3	$1 + 3\lambda$		
4	λ^4	$1 + 4\lambda + 6\lambda^2$	3	
5	λ^5	$1 + 5\lambda + 10\lambda^2 + 10\lambda^3$	$10 + 15\lambda$	
6	λ^6	$1 + 6\lambda + 15\lambda^2 + 20\lambda^3 + 15\lambda^4$	$25 + 60\lambda + 45\lambda^2$	15

It follows from (2.14) that

$$k! \sum_{n=0}^{\infty} Q(n, k, \lambda)\frac{x^n}{n!} = e^{\lambda x}(e^x - x - 1) . \tag{2.18}$$

By comparing coefficients of x^n on both sides of (2.18), we get an explicit formula for $Q(n, k, \lambda)$:

$$Q(n, k, \lambda) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{t=0}^{k-j}\binom{k-j}{t}(n)_t(\lambda + j)^{n-t}, \tag{2.19}$$

where $(n)_t = n(n-1) \ldots (n-t+1)$.

It follows from Theorem 2.1 and the generating function for $R(n, k, \lambda)$ that

$$R(n, n-k, \lambda) = \sum_{j=0}^{k} Q(2k-j, k-j, \lambda)\binom{n}{2k-j}, \tag{2.20}$$

which shows that $R(n, n-k, \lambda)$ is a polynomial in n of degree $2k$. Equation (2.20) also shows that $R'(n, k, \lambda) = Q(2n-k, n-k, \lambda)$, where $R'(n, k, \lambda)$ is defined by Carlitz in [2].

In [1], Carlitz generalized the Bell number [4, p. 210] by defining

$$B(n, \lambda) = \sum_{k=0}^{n} R(n, k, \lambda). \tag{2.21}$$

This suggests the definition

$$A(n, \lambda) = \sum_{k=0}^{n} Q(n, k, \lambda), \tag{2.22}$$

which for $\lambda = 0$ reduces to

$$A(n) = \sum_{k=0}^{n} b(n, k).$$

The function $A(n)$ appears in [5] and [6].

By (2.13), we have

$$A(n, \lambda) = \sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{n-m} b(n-m, k)\lambda^m = \sum_{m=0}^{n}\binom{n}{m}\lambda^m A(n-m). \tag{2.23}$$

Also by (2.18),

$$\sum_{n=0}^{\infty} A(n, \lambda)\frac{x^n}{n!} = e^{\lambda x}\exp(e^x - x - 1), \tag{2.24}$$

and (2.24) implies

$$\sum_{n=0}^{\infty} A(n, \lambda)\frac{x^n}{n!} = e^{x(\lambda-1)}\exp(e^x - 1) = \sum_{n=0}^{\infty} B(n, \lambda - 1)\frac{x^n}{n!},$$

so

$$A(n, \lambda) = B(n, \lambda - 1). \tag{2.25}$$

For example, $A(n, 1) = B(n, 0)$, so

$$\sum_{k=0}^{[(n+1)/2]} (b(n, k) + b(n + 1, k)) = \sum_{k=0}^{n} S(n, k).$$

There are combinatorial interpretations of $A(n, \lambda)$ and $Q(n, k, \lambda)$ that are similar to the interpretations of $B(n, \lambda)$ and $R(n, k, \lambda)$ given in [1]. Let λ be a nonnegative integer and let $B_1, B_2, \ldots, B_\lambda$ denote λ open boxes. Let $P(n, k, \lambda)$ denote the number of partitions of Z_n into k blocks with each block containing at least two elements, with the understanding that an arbitrary number of the elements of Z_n may be placed in any number (possibly none) of the boxes. We shall call these λ_1 partitions. Clearly, $P(n, k, 0) = b(n, k)$.

Now, if i elements are placed in the λ boxes, there are $\binom{n}{i}$ ways to choose the elements, and for each element chosen there are λ choices for a box. The number of such partitions is $\binom{n}{i}\lambda^i b(n - i, k)$. Hence,

$$P(n, k, \lambda) = \sum_{m=0}^{n} \binom{n}{m}\lambda^m b(n - m, k) = Q(n, k, \lambda). \tag{2.26}$$

It is clear from (2.26) that $A(n,\lambda)$ is the number of λ_1 partitions of Z_n.

It is also clear from (2.7) and the above comments that $\overline{b}(n, k + 1, \lambda)$ is the number of λ_1 partitions of Z_n into k blocks such that at least two elements of Z_n are placed in the open boxes. Definition (2.4) furnishes another combinatorial interpretation of $\overline{b}(n, k, \lambda)$.

Finally, we note that some of the definitions and formulas in this section can be generalized in terms of the r-associated Stirling numbers of the second kind $b_r(n, k)$. These numbers are defined by means of

$$\left(e^x - \sum_{i=0} \frac{X^i}{i!}\right)^k = k!\sum_{n=0}^{\infty} b_r(n, k)\frac{X^n}{n!},$$

and their properties are examined in [3], [5], and [6]. Using the methods of this section, we can define functions $\overline{b}_r(n, k, \lambda)$, $Q^{(r)}(n, k, \lambda)$ and $A^{(r)}(n, \lambda)$ which reduce to $\overline{S}(n, k, \lambda)$, $R(n, k, \lambda)$, and $B(n, \lambda)$ when $r = 0$, and reduce to $\overline{b}(n, k, \lambda)$, $Q(n, k, \lambda)$, and $A(n, \lambda)$ when $r = 1$. The combinatorial interpretations and formulas (2.4)-(2.7), (2.10), (2.11), (2.17), (2.18), (2.22), (2.23) can all be generalized.

3. THE FUNCTIONS $d(n, k, \lambda)$ AND $Q_1(n, k, \lambda)$

We define $\langle\lambda\rangle_j = \lambda(\lambda + 1) \ldots (\lambda + j - 1)$. Now put

$$d(n;\ k_2,\ \ldots,\ k_n;\ \lambda) = \sum\left(k_2 \frac{\langle\lambda\rangle_2}{1!} + \cdots + k_n \frac{\langle\lambda\rangle_n}{(n-1)!}\right), \tag{3.1}$$

where the summation is over all permutations of Z_n,

$$n = 2k_2 + 3k_3 + \cdots + nk_n,$$

with k_2 cycles of length 2, k_3 cycles of length 3, ..., k_n cycles of length n. Then, as in [1], we sum on both sides of (3.1) and obtain, after some manipulation,

$$\begin{aligned}&\sum_{n=2}^{\infty} \frac{x^n}{n!} \sum_{k_2, k_3, \ldots} d(n;\ k_2,\ k_3,\ \ldots;\ \lambda)y^k \\ &= y((1 - x)^{-\lambda} - \lambda x - 1)\exp\{y(-\log(1 - x) - x)\}.\end{aligned} \tag{3.2}$$

We now define

$$\bar{d}(n,\ k,\ \lambda) = \sum\sum\left(k_2 \frac{\langle\lambda\rangle_2}{1!} + k_3 \frac{\langle\lambda\rangle_3}{2!} + \cdots + k_n \frac{\langle\lambda\rangle_n}{(n-1)!}\right), \tag{3.3}$$

where the inner summation is over all permutations of Z_n with k_2 cycles of length 2, k_3 cycles of length 3, ..., k_n cycles of length n; the outer summation is over all $k_2,\ k_3,\ \ldots,\ k_n$ satisfying (2.1).

By (3.2) and (3.3), we have

$$\begin{aligned}\sum_{n,k} \bar{d}(n,\ k,\ \lambda)\frac{x^n}{n} y^k &= y((1 - x)^{-\lambda} - \lambda x - 1)\exp\{y(-\log(1 - x) - x)\} \\ &= y((1 - x)^{-\lambda} - \lambda x - 1)(1 - x)^{-y}e^{-xy},\end{aligned} \tag{3.4}$$

and from (3.4), we obtain

$$k!\sum_{n=0}^{\infty} \bar{d}(n,\ k + 1,\ \lambda)\frac{x^n}{n} = ((1 - x)^{-\lambda} - \lambda x - 1)(-\log(1 - x) - x)^k. \tag{3.5}$$

It follows from (1.2) and (3.5) that

$$\bar{d}(n,\ k,\ \lambda) = \sum_{m=2}^{n-2k+m} \binom{n}{m} d(n - m,\ k - 1)\langle\lambda\rangle_m. \tag{3.6}$$

For $\lambda = 1$, (3.4) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{d}(n,\ k,\ 1)\frac{x^n}{n} y &= y((1 - x)^{-1} - x - 1)\exp\{y(-\log(1 - x) - x)\} \\ &= \frac{\partial}{\partial x} \exp\{y(-\ln(1 - x) - x)\} - xy \exp\{(y(-\log(1 - x) - x)\} \\ &= \sum_{n,k} d(n + 1,\ k)\frac{x^n}{n!} y^k - \sum_{n,k} nd(n - 1,\ k - 1)\frac{x^n}{n!} y^k.\end{aligned}$$

Thus, we have

$$\bar{d}(n,\ k,\ 1) = d(n + 1,\ k) - nd(n - 1,\ k - 1) = nd(n,\ k). \tag{3.7}$$

We also have, by (3.5) and (3.6),

$$\bar{d}(n,\ 0,\ \lambda) = 0$$

$$\bar{d}(n,\ 1,\ \lambda) = \langle\lambda\rangle_n \text{ if } n \geqslant 2,$$

$$\bar{d}(n,\ 2,\ \lambda) = \binom{n}{2}(n-3)!\langle\lambda\rangle_2 + \binom{n}{3}(n-4)!\langle\lambda\rangle_3 + \cdots + \binom{n}{n-2}1!\langle\lambda\rangle_{n-2},$$

$$\bar{d}(n,\ k,\ \lambda) = 0 \text{ if } n < 2k,$$

$$\bar{d}(2k,\ k,\ \lambda) = \binom{2k}{2}d(2k-2,\ k-1)\langle\lambda\rangle_2.$$

To find the relationship to $\overline{S}_1(n,\ k,\ \lambda)$, we use Theorem (2.1). We define $\bar{c}(n,\ k,\ \lambda)$ by

$$((1-x)^{-\lambda} - 1)(-\log(1-x) - x)^k = k!\sum_{n=0}^{\infty}\bar{c}(n,\ k+1,\ \lambda)\frac{x^n}{n!}. \tag{3.8}$$

Then by Theorem 2.1 and the generating function for $\overline{S}_1(n,\ k,\ \lambda)$,

$$\overline{S}_1(n,\ n-k,\ \lambda) = \sum_{j=0}^{k}\binom{n}{2k-j+1}\bar{c}(2k-j+1,\ k-j+1,\ \lambda). \tag{3.9}$$

By (3.5) and (3.8),

$$\bar{c}(n,\ k+1,\ \lambda) = \bar{d}(n,\ k+1,\ \lambda) + \lambda n d(n-1,\ k),$$

so by (1.3), equation (3.9) can be written

$$\begin{aligned}\overline{S}_1(n,\ n-k,\ \lambda) = \sum_{j=0}^{k}\bar{d}(2k-j+1,\ k-j+1,\ \lambda)\binom{n}{2k-j+1}\\ + \lambda n S_1(n-1,\ n-1-k),\end{aligned} \tag{3.10}$$

which proves $\overline{S}_1(n,\ n-k,\ \lambda)$ is a polynomial in n of degree $2k+1$.

We now define the function $Q_1(n,\ k,\ \lambda)$ by means of

$$Q_1(n,\ k,\ \lambda) = \bar{d}(n,\ k+1,\ \lambda) + d(n,\ k) + n\ d(n-1,\ k). \tag{3.11}$$

then by (3.6),

$$Q_1(n,\ k,\ \lambda) = \sum_{m=0}^{n-2k}\binom{n}{m}d(n-m,\ k)\langle\lambda\rangle_m. \tag{3.12}$$

Note that $Q_1(n,\ k,\ 0) = d(n,\ k)$.

A generating function can be found by summing on both sides of (3.11). We have

$$\begin{aligned}\sum_{n,k} Q_1(n,\ k,\ \ )\frac{x^n}{n}\ y &= (1-x)^{-\lambda}\exp\{y(-\log(1-x) - x)\}\\ &= (1-x)^{-\lambda - y}e^{-xy}.\end{aligned} \tag{3.13}$$

If we differentiate (3.13) with respect to x, multiply by $1-x$, and then compare coefficients of $x^n y^k$, we obtain

$$Q_1(n+1,\ k,\ \lambda) = (\lambda + n)Q_1(n,\ k,\ \lambda) + nQ_1(n-1,\ k-1,\ \lambda). \tag{3.14}$$

If we multiply both sides of (3.13) by $1-x$ and compare coefficients $x^n y^k$, we have

$$Q_1(n, k, \lambda - 1) = Q_1(n, k, \lambda) - nQ_1(n - 1, k, \lambda). \tag{3.15}$$

For $\lambda = 1$, (3.14) and (3.15) can be combined to yield

$$d(n + 1, k + 1) = nQ_1(n - 1, k, 1). \tag{3.16}$$

Also, if $\lambda = 0$ in (3.15), we have

$$Q_1(n, k, -1) = d(n, k) - nd(n - 1, k).$$

In addition

$$Q_1(n, 0, \lambda) = \langle\lambda\rangle_n,$$

$$Q_1(n, 1, \lambda) = (n - 1)! + \binom{n}{1}(n - 2)!\langle\lambda\rangle_1 + \cdots + \binom{n}{n-2}1!\langle\lambda\rangle_{n-2},$$

$$Q_1(n, k, 0) = d(n, k),$$

$$Q_1(2k, k, \lambda) = d(2k, k),$$

$$Q_1(n, k, \lambda) = 0 \text{ if } n < 2k.$$

A small table of values is given below.

$Q_1(n, k, \lambda)$

$n \backslash k$	0	1	2	3
0	1			
1	λ			
2	$\langle\lambda\rangle_2$	1		
3	$\langle\lambda\rangle_3$	$2 + 3\lambda$		
4	$\langle\lambda\rangle_4$	$6 + 14\lambda + 6\lambda^2$	3	
5	$\langle\lambda\rangle_5$	$24 + 70\lambda + 50\lambda^2 + 10\lambda^3$	$20 + 15\lambda$	
6	$\langle\lambda\rangle_6$	$120 + 404\lambda + 375\lambda^2 + 130\lambda^3 + 15\lambda^4$	$130 + 65\lambda + 45\lambda^2$	15

It follows from (3.13) that

$$k!\sum_{n=0}^{\infty} Q_1(n, k, \lambda)\frac{x^n}{n!} = (1 - x)^{-\lambda}(-\log(1 - x) - x)^k, \tag{3.17}$$

and from Theorem 2.1, that

$$R_1(n, n - k, \lambda) = \sum_{j=0}^{k} Q_1(2k - j, k - j, \lambda)\binom{n}{2k - j}, \tag{3.18}$$

which shows that $R_1(n, n - k, \lambda)$ is a polynomial in n of degree $2k$. Equation (3.18) also shows that $R'(n, k, \lambda) = Q_1(2n - k, n - k, \lambda)$, where $R_1'(n, k, \lambda)$ is defined by Carlitz in [2].

Letting $y = 1$ in (3.13), we have

$$\sum_{k=0}^{[n/2]} Q_1(n, k, \lambda) = \sum_{t=0}^{n} (-1)^{n-t}\binom{n}{t}\langle\lambda + 1\rangle_t,$$

and more generally,

$$\sum_{k=0}^{[n/2]} Q_1(n, k, \lambda)y^k = \sum_{t=0}^{n} (-y)^{n-t}\binom{n}{t}\langle \lambda + y \rangle_t .$$

A combinatorial interpretation of $Q_1(n, k, \lambda)$ follows. Let λ be a nonnegative integer and let $B_1, B_2, \ldots, B_\lambda$ denote λ open boxes. Let $P_1(n, k, \lambda)$ denote the number of permutations of Z_n with k cycles such that each cycle contains at least two elements, with the understanding that an arbitrary number of elements of Z_n may be placed in any number (possibly none) of the boxes and then permuted in all possible ways in each box. We call these λ_1 permutations. Clearly, $P_1(n, k, 0) = d(n, k)$.

If i elements are placed in the boxes, there are $\binom{n}{i}$ ways to choose the elements and then $\lambda(\lambda + 1)\ (\lambda + 2) \ldots (\lambda + i - 1)$ ways to place the elements in the boxes. The number of such permutations is $\binom{n}{i}\langle \lambda \rangle_i d(n - i, k)$. Hence,

$$P_1(n, k, \lambda) = \sum_{m=0}^{n} \binom{n}{m}\langle \lambda \rangle_m d(n - m, k) = Q_1(n, k, \lambda). \tag{3.19}$$

It is clear from (3.6) and the above comments that $\bar{d}(n, k + 1, \lambda)$ is the number of λ_1 permutations of Z_n with k cycles such that at least two elements of Z_n are placed in the open boxes. Definition (3.3) furnishes another combinatorial interpretation of $\bar{d}(n, k, \lambda)$.

We note that some of the definitions and formulas in this section can be generalized in terms of the r-associated Stirling numbers of the first kind $d_r(n, k)$. These numbers are defined by means of

$$\left(-\log(1 - x) - \sum_{i=1}^{r} \frac{X^i}{i!}\right)^k = k! \sum_{n=0}^{\infty} d_r(n, k)\frac{X^n}{n!},$$

and their properties are discussed in [3] and [6]. Using the methods of this section, we can define functions $d_r(n, k, \lambda)$ and $Q^{(r)}(n, k, \lambda)$ which reduce to $\bar{S}_1(n, k, \lambda)$ and $R_1(n, k, \lambda)$ when $r = 0$, and to $\bar{d}(n, k, \lambda)$ and $Q_1(n, k, \lambda)$ when $r = 1$. The combinatorial interpretations and formulas (3.3)-(3.6), (3.11)-(3.14), and (3.17) can all be generalized.

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