DIGIT FUNCTIONS OF INTEGER SEQUENCES

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1. SHIFTED INTEGER SEQUENCES

It was noticed by Benford [1] that the first nonzero digit in certain sets of real numbers is not uniformly distributed among the integers 1 through 9; in fact, the probability that this first, leftmost digit equals β is equal to

$$\log_{10}(1 + \beta^{-1}).$$

He extended the analysis to the frequency of digits beyond the first for numbers obeying a particular probability law: the logarithmic distribution. This phenomenon of nonuniform distribution of digits has generated a considerable mathematical literature, particularly for the first digit, and has been shown to apply to the Fibonacci numbers [2], [3], [4].

The purpose of this paper is to examine the probabilistic structure of the entire set of digits from certain integer sequences. The Fibonacci sequence provides one example.

The essential results are that, for a large class of probability laws, the digits are not equiprobable and their values are correlated; but in the limit, as the ordinal number of the digits goes to infinity, the digit values approach equiprobability and their correlation goes to zero. However, under certain conditions, this limiting behavior does not occur; rather, the nonuniform behavior persists for all digits. In particular, subsequences of the Fibonacci sequence exist which exhibit "persistent Benford" behavior.

Let $\omega = \{a_n\}$ be a sequence of positive integers. Define a shifted sequence $\hat{\omega}$ of rationals $\hat{a}_n \in U_b = [b^{-1}, 1]$, for integer base $b \geqslant 2$, by

$$\hat{a}_n = a_n b^{-v(a_n)}$$

where

$$v(a_n) = [\log_b a_n] + 1$$

is the number of digits in the b-adic representation of a_n, with $[\cdot]$ the greatest integer function.

The asymptotic distribution function (a.d.f.) $g : U_b \to E^1$ is defined for $\hat{\omega}$ as usual by

$$g(x) = \lim_{N \to \infty} \frac{A([b^{-1}, x);\ N;\ \hat{\omega})}{N} \tag{1}$$

when this limit exists. Here A is the counting function which records the number among the first N terms of $\hat{\omega}$ that lie in the interval $[b^{-1}, x)$. Note that g is left-continuous.

Theorem 1: If $a_n = \alpha^n$, $\alpha > 1$ and not a rational power of b, then the a.d.f. g of $\{\hat{a}_n\}$ exists and

$$g(x) = 1 + \log_b x. \tag{2}$$

Proof: Since $\hat{a}_n \leqslant x$ if and only if $1 + \log_b \hat{a}_n \leqslant 1 + \log_b x$,

$$g(x) = \lim_{N \to \infty} \frac{A([0,\ 1 + \log_b x);\ N;\ \{1 + \log_b \hat{a}_n\})}{N}$$

if the limit exists. But, since α is not a rational power of b,

$$\{1 + \log_b \hat{a}_n\} = \{1 + n\xi\},\ \xi \text{ irrational},$$

is uniformly distributed mod 1, thus yielding the theorem. ■

It can be shown that (2) is also the a.d.f. of the shifted sequence $\{\hat{F}_n\}$ of Fibonacci numbers F_n because

$$F_n \sim \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^{n+1}$$

(see also [5]). In fact, this a.d.f. holds for any integer sequence defined by a recurrence relation.

An example of an important sequence of integers that does not have an a.d.f. is the sequence of primes. It was shown by Wintner [6] that the limit (1) does not exist in this case. However, the relative logarithmic density does exist [7].

Theorem 2: If $\{\hat{a}_n\}$ has a continuous a.d.f. g, then for every Riemann-integrable function $f : U_b \to E^1$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(\hat{a}_n) = \int_{b^{-1}}^{1} f(x)\,dg(x).$$

Proof: Immediate from Theorem 7.2 of [8]. ■

Theorem 2 provides us the means to apply the standard facts of probability theory to the study of digit functions of integer sequences.

2. DIGIT FUNCTIONS AND ASYMPTOTIC EQUIPROBABILITY

Let the digit function d_k be defined such that $d_k(x)$ equals the k^{th} digit of x so that

$$x = \sum_{k=1}^{\infty} d_k(x) b^{-k}.$$

Define

$$T[\beta(k)] = \{x \in U_b \mid d_k(x) = \beta(k)\} \subseteq U_b,$$

where $\beta(k) \in Z_b = \{0,\ \ldots,\ b - 1\}$. Then, the joint probability p_g that

$$d_{k_1}(x) = \beta(k_1),\ \ldots,\ d_{k_s}(x) = \beta(k_s)$$

is given by the Lebesgue-Stieltjes integral

$$p_g[\beta(k_1),\ \ldots,\ \beta(k_s)] = \int_{b^{-1}}^{1} I_{T[\beta(k_1)]} \cdots I_{T[\beta(k_s)]}\,dg(x), \tag{3}$$

where I_G is the indicator function of the set $G \subseteq U_b$. Allowing some abuse of notation, the same symbol p_g will be used for all such probability functions, regardless of the dimensionality of the domain. Also, when no confusion will result, the argument k of β will be suppressed.

When g is the logarithmic distribution (2),

$$p_g[\beta(k_1), \ldots, \beta(k_s)] = \sum_{\beta(1)=1}^{b-1} \sum_{\beta(2)=0}^{b-1} \cdots \sum_{\beta(k_s-1)=0}^{b-1} \log_b \left[1 + \frac{b^{-k_s}}{\sum_{m=1}^{k_s} \beta(m) b^{-m}}\right], \tag{4}$$

where the sums over $\beta(k_j)$ for $j = 1, \ldots, s - 1$ are to be excluded.

The relative frequency of digit values will be derived by setting $s = 1$ in (3) and (4). The succeeding section uses $s = 2$ to infer dependence properties between digits.

Definition 1: The a.d.f. g is *asymptotically equiprobable* with respect to b if and only if

$$\lim_{k \to \infty} p_g[\beta(k)] = b^{-1} \text{ for all } \beta \in Z_b. \tag{5}$$

It can be shown that g is asymptotically equiprobable if a density function f exists for g. Furthermore, for a sufficiently smooth a.d.f., such as the logarithmic distribution (2), the rate of approach can also be displayed, as in Theorem 3. When f exists, p_g and p_f will be used interchangeably to denote the function defined in (3), as suits the occasion, with the symbol f being reserved for the density function and g for the a.d.f.

Theorem 3: If $f \in C^2[b^{-1}, 1]$, then

$$p_f[\beta(k)] = b^{-1} + h(\beta)b^{-k} + 0(b^{-2k}) \text{ for all } \beta \in Z_b,$$

where

$$h(\beta) = \frac{1}{2}\left(\frac{2\beta + 1}{b} - 1\right)[f(1) - f(b^{-1})].$$

Proof: Let $q_i[\beta(k)]$ be the b-adic rationals defined by

$$T[\beta(k)] = \bigcup_{i=1}^{M} [q_i[\beta(k)], q_i[\beta(k)] + b^{-k}] \tag{6}$$

with

$$M = \begin{cases} 1, & k = 1, \\ (b - 1)b^{k-2}, & k > 1. \end{cases} \tag{7}$$

Then, writing q_i for $q_i[\beta(k)]$,

$$\begin{aligned} p_f[\beta(k)] &= \int_{b^{-1}}^{1} I_{T[\beta(k)]} f(x)\,dx = \sum_{i=1}^{M} \int_{q_i}^{q_i + b^{-k}} f(x)\,dx \\ &= \sum_{i=1}^{M} \frac{1}{2} b^{-k} [f(q_i) + f(q_i + b^{-k})] + 0(b^{-2k}), \end{aligned}$$

where the last equality follows from the trapezoidal rule of integration [9]. The two ordinate sums in this last equation can be converted into integrals, with remainders, by use of the Euler-Maclaurin formula [10]. For $k > 1$,

$$\sum_{i=1}^{M} \frac{1}{2} b^{-k} f(q_i) = \frac{1}{2b} \sum_{i=1}^{M} b^{-k+1} f[b^{-1} + (i - 1)b^{-k+1} + \beta b^{-k}]$$
$$= \frac{1}{2b} \int_{b^{-1}}^{1} f(x)dx + \frac{b^{-k+1}}{2b}\left(\frac{\beta}{b} - \frac{1}{2}\right)[f(1) - f(b^{-1})].$$

For $k = 1$, $q_i = \beta b^{-1}$, and the same result is obtained. Calculating a similar expression for the term involving $f(q_i + b^{-k})$ and using the fact that

$$\int_{b^{-1}}^{1} f(x)dx = 1$$

yield the theorem. ■

Using Theorem 3, the expected value of the k^{th} digit of x is

$$E(d_k) = \frac{b - 1}{2} + b^{-k}[f(1) - f(b^{-1})]\frac{b^2 - 1}{12} + 0(b^{-2k}),$$

which is approximately $(b - 1)/2$ for large k (as expected!).

To denote the special case of the density function corresponding to the logarithmic distribution (applicable to the Fibonacci sequence), r will be used in place of f; that is,

$$r(x) = \frac{d \log_b(x)}{dx} = \frac{1}{x \ln b},$$

which has been termed the "reciprocal density function" [11]. Theorem 3 applies and gives

$$p_r[\beta(k)] = b^{-1} + h(\beta)b^{-k} + 0(b^{-2k}).$$

Theorem 4:

$$p_r[\beta(k)] = \sum_{i=1}^{M} \log_b\left(1 + \frac{b^{-k}}{q_i}\right),$$

where q_i is defined by (6) and M by (7).

Proof:

$$p_r[\beta(k)] = \int_{b^{-1}}^{1} I_{T[\beta(k)]} r(x)dx = \sum_{i=1}^{M} \int_{q_i}^{q_i + b^{-k}} \frac{dx}{x \ln b}$$
$$= \sum_{i=1}^{M} \frac{1}{\ln b}[\ln(q_i + b^{-k}) - \ln(q_i)],$$

which yields the theorem. ■

For the special case $b = 10$, the relative frequencies, obtained from Theorem 4, of values of the first four digits are given in the accompanying table. The last digit in each entry has been rounded and not truncated. Columns 1 and 2 contain Benford's original results. For subsequent digits, the rapid approach of these data to b^{-1} is readily apparent when plotted as in Figure 1.

Figure 2 provides samples of the convergence of the relative frequency of second-digit values for the Fibonacci sequence to their theoretical limits (cf.

column 2 of the table). The fraction of the first N Fibonacci numbers with second digit equal to β is plotted against N for five values of β.

Probability that Digit k Equals β for the Logarithmic Distribution (Base 10)

β \ k	1	2	3	4
0	-	.11968	.10178	.10018
1	.30103	.11389	.10138	.10014
2	.17609	.10882	.10097	.10010
3	.12494	.10433	.10057	.10006
4	.09691	.10031	.10018	.10002
5	.07918	.09668	.09979	.09998
6	.06695	.09337	.09940	.09994
7	.05799	.09035	.09902	.09990
8	.05115	.08757	.09864	.09986
9	.04576	.08500	.09827	.09982

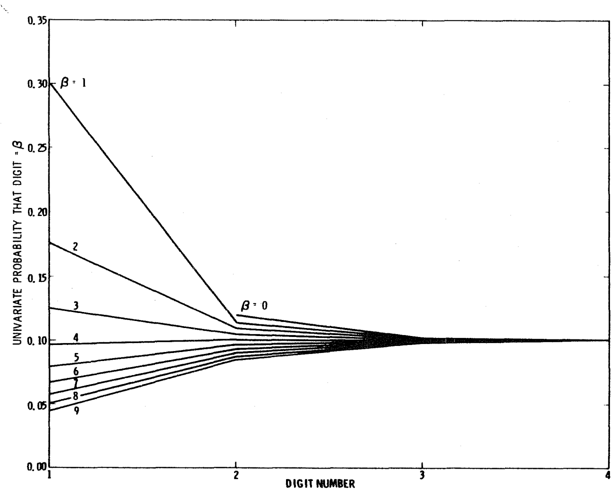

Fig. 1. Approach of Relative Frequency of Digits to b^{-1}. Logarithmic Distribution with $b = 10$

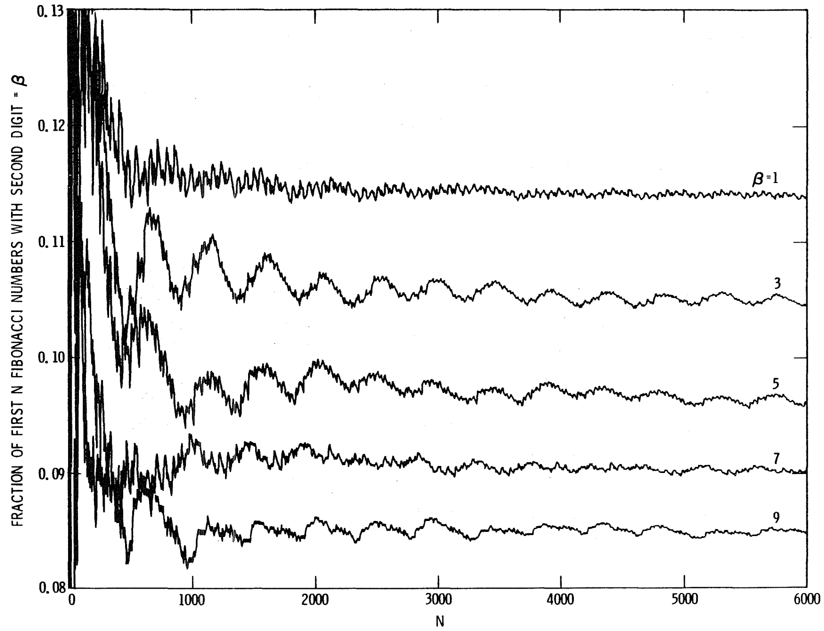

Fig. 2. Convergence of Relative Frequencies to Theoretical Values for Second Digit of Fibonacci Numbers

There exist integer sequences for which asymptotic equiprobability does not hold (for the a.d.f.). For example, Benford's first-digit frequencies can be retained for all subsequent digits for certain subsequences of the Fibonacci sequence, and, in the next theorem, conditions are given for the existence of integer sequences which possess specified digit properties, a reversal of the approach used thus far.

Theorem 5: For each $k = 1, 2, \ldots$, let t_k be a function from the Cartesian product of Z_b with itself k times to $[0, 1]$ and satisfying the three consistency conditions:

$$t_k[\beta(1), \ldots, \beta(k)] \geqslant 0; \qquad \sum_{\beta(1) \in Z_b} t_1[\beta(1)] = 1;$$

and

$$\sum_{\beta(k+1) \in Z_b} t_{k+1}[\beta(1), \ldots, \beta(k), \beta(k+1)] = t_k[\beta(1), \ldots, \beta(k)].$$

Then, for any integer sequence ω with $\hat{\omega}$ dense in U_b, there exists a subsequence τ with a.d.f. g such that $p_g = t_k$.

Proof: By Billingsley's theorem [12] (a consequence of Kolmogorov's existence theorem), the three conditions on t_k insure the existence of a probability measure μ on the Borel sets of U_b such that, for each k,

$$\mu(T[\beta(1)] \cap \cdots \cap T[\beta(k)]) = t_k[\beta(1), \ldots, \beta(k)]$$

for all $\beta(1), \ldots, \beta(k)$ in Z_b.

Then define a distribution $g : U_b \to [0, 1]$ by $g(x) = \mu[b^{-1}, x)$. By Theorem 4.3 of [8], there exists a sequence $\hat{\sigma}$ in U_b with a.d.f. $g_\sigma = g$.

Let $\hat{\sigma} = \{s_j\}$. Since $\hat{\omega}$ is dense in U_b, there exists a subsequence τ of ω with $\hat{\tau} = \{v_j\}$ such that $v_j = s_j + \Delta_j$, where $\Delta_j \geqslant 0$ and $\lim\limits_{j \to \infty} \Delta_j = 0$.

Since $\Delta_j \geqslant 0$,

$$A([b^{-1}, x); N; \hat{\tau}) \leqslant A([b^{-1}, x); N; \hat{\sigma}). \tag{8}$$

For $\varepsilon > 0$, choose N_0 such that $\Delta_j < \varepsilon$ for $j \geqslant N_0$. Then

$$A([b^{-1}, x_\varepsilon); N; \{s_j\}_{N_0}^{\infty}) \leqslant A([b^{-1}, x); N; \{v_j\}_{N_0}^{\infty}), \tag{9}$$

where $x_\varepsilon = \min\{b^{-1}, x - \varepsilon\}$.

By (1), there exists k_N such that

$$\frac{A([b^{-1}, x); N; \{s_j\}_{N_0}^{\infty})}{N} = g_\sigma(x) + k_N(x),$$

where $\lim\limits_{N \to \infty} k_N(x) = 0$ for every $x \in U_b$.

Using (8) and (9):

$$g_\sigma(x - \varepsilon) - g_\sigma(x) + k_N(x - \varepsilon) \leqslant \frac{A([b^{-1}, x); N; \{v_j\}_{N_0}^{\infty})}{N} - g_\sigma(x) \leqslant k_N(x).$$

Letting N go to ∞ gives

$$g_\sigma(x - \varepsilon) - g_\sigma(x) \leqslant g_\tau(x) - g_\sigma(x) \leqslant 0.$$

Since g_σ is continuous from the left and ε is arbitrary, $g_\tau = g_\sigma = g$, and the theorem is established. ∎

Definition 2: An integer sequence ω is said to be *absolutely equiprobable* with respect to b if and only if

$$\lim_{N \to \infty} \frac{A(T[\beta(k)]; N; \hat{\omega})}{N} = \begin{cases} (b - 1)^{-1}, & k = 1 \\ b^{-1}, & k > 1 \end{cases}, \text{ for all } \beta \in Z_b.$$

Corollary 5.1: For every $b \geqslant 2$, there exists a subsequence of the Fibonacci numbers that is absolutely equiprobable with respect to b.

Proof: Let $t_k[\beta(1), \ldots, \beta(k)] = (b - 1)^{-1}b^{-k+1}$. Then, by Theorem 5, there exists a subsequence τ of $\{F_n\}$ with a.d.f. g such that $p_g = t_k$ for all k.

Since

$$A(T[\beta(k)]; N; \hat{\tau}) = \sum_i A([q_i, q_i + b^{-k}); N; \hat{\tau}),$$

then

$$\begin{aligned}\lim_{N \to \infty} \frac{A(T[\beta(k)]; N; \hat{\tau})}{N} &= \sum_i [g(q_i + b^{-k}) - g(q_i)] = p_g(\beta(k)) \\ &= \sum{}' p_g(\beta(1), \ldots, \beta(k - 1), \beta(k)) \\ &= \sum{}' t_k(\beta(1), \ldots, \beta(k - 1), \beta(k)) \\ &= b^{-1}\sum{}' t_{k-1}(\beta(1), \ldots, \beta(k - 1)),\end{aligned}$$

where Σ' denotes the sum over all $\beta(j)$ for $j < k$. Then, $k - 2$ applications of the third consistency condition of Theorem 5, followed by use of the second condition, yields the corollary. The case $k = 1$ is trivial. Thus,

$$\lim_{N \to \infty} \frac{A(T[\beta(k)];\ N;\ \hat{\tau})}{N} = b^{-1}$$

as required. ■

Definition 3: An integer sequence ω is said to be a *persistent Benford sequence* with respect to b if and only if

$$\lim_{N \to \infty} \frac{A(T[\beta(k)];\ N;\ \hat{\omega})}{N} = \begin{cases} \log_b(1 + \beta^{-1}(k)), & \beta(k) > 0 \\ 0, & \beta(k) = 0, \end{cases}$$

for all $k \geqslant 1$ and all $\beta(k) \in Z_b$.

Corollary 5.2: For every $b \geqslant 2$, there exists a subsequence of the Fibonacci numbers that is persistent Benford with respect to b.

Proof: A calculation similar to that contained in the proof of Corollary 5.1 serves here and, in fact, for any t_k defined as the product of univariate density functions. ■

3. WEAK DEPENDENCE OF DIGIT FUNCTIONS

Dependence between digit functions is demonstrated by showing that they are correlated random variables.

First, an expression for the bivariate density function is derived.

Theorem 6: If $f \in C^2[b^{-1}, 1]$ and $k_2 > k_1$, then

$$p_f[\beta(k_1),\ \beta(k_2)] = b^{-1}p_f[\beta(k_1)] + h[\beta(k_2)]b^{-k_2-1} + \tilde{h}[\beta(k_1),\ \beta(k_2)]b^{-k_1-k_2} + O(b^{-\min\{2k_1+k_2,\ 2k_2\}})$$

where the function h is defined in Theorem 3 and

$$\tilde{h}[\beta(k_1),\ \beta(k_2)] = \frac{b}{4}\left[B_1\left(\frac{\beta(k_2)}{b}\right) + B_1\left(\frac{\beta(k_2) + 1}{b}\right)\right] \times \left[B_2\left(\frac{\beta(k_1) + 1}{b}\right) - B_2\left(\frac{\beta(k_1)}{b}\right)\right][f'(1) - f'(b^{-1})]$$

with B_1, B_2 Bernoulli polynomials and the prime denoting differentiation.

Proof: Let $u_i(\beta(k_1),\ \beta(k_2))$ be the b-adic rationals defined by

$$T[\beta(k_1)] \cap T[\beta(k_2)] = \bigcup_{i=1}^{ML} [u_i(\beta(k_1),\ \beta(k_2)),\ u_i(\beta(k_1),\ \beta(k_2)) + b^{-k_2}],$$

where M is defined in (7), $L = b^{k_2-k_1-1}$ and $i = (i_1 - 1)L + i_2$. Then, writing u_i for $u_i(\beta(k_1),\ \beta(k\ ))$,

$$p_f[\beta(k_1),\ \beta(k_2)] = \sum_{i_1=1}^{M} \sum_{i_2=1}^{L} \int_{u_i}^{u_i+b^{-k_2}} f(x)dx.$$

Using the trapezoidal rule,

$$p_f[\beta(k_1),\ \beta(k_2)] = \sum_{i_1=1}^{M}\sum_{i_2=1}^{L}\frac{b^{-k_2}}{2}[f(u_i) + f(u_i + b^{-k_2})] + O(b^{-3k_2}).$$

Substituting $u_i = q_{i_1} + (i_2 - 1)b^{-k_2+1} + \beta(k_2)b^{-k_2}$ in this expression and applying the Euler-Maclaurin formula, as in the proof of Theorem 3, to the sums over i_2 gives

$$\begin{aligned}p_f[\beta(k_1),\ \beta(k_2)] &= \frac{1}{b}\sum_{i_1=1}^{M}\int_{q_{i_1}}^{q_{i_1}+b^{-k_1}} f(x)dx + \frac{1}{2b}\sum_{i_1=1}^{M} b^{-k_2+1}\\ &\times\left[\left(B_1\left(\frac{\beta(k_2)}{b}\right) + B_1\left(\frac{\beta(k_2)+1}{b}\right)\right)[f(q_{i_1} + b^{-k_1}) - f(q_{i_1})]\right.\\ &+ \frac{b^{-k_2+1}}{2}\left(B_2\left(\frac{\beta(k_2)}{b}\right) + B_2\left(\frac{\beta(k_2)+1}{b}\right)\right)\\ &\left.\times\ [f'(q_{i_1} + b^{-k_1}) - f'(q_{i_1})]\right] + O(b^{-k_1-3k_2}).\end{aligned}$$

Recognizing the univariate expression for digit k_1 in the first term and again applying the Euler-Maclaurin formula to each of the four sums inherent in the second term yields

$$\begin{aligned}p_f[\beta(k_1),\ \beta(k_2)] &= \frac{1}{b}\,p_f[\beta(k_1)] + \frac{1}{2b}\,b^{-k_2+1}\left[B_1\left(\frac{\beta(k_2)}{b}\right) + B_1\left(\frac{\beta(k_2)+1}{b}\right)\right]\\ &\times\left[\left[B_1\left(\frac{\beta(k_1)+1}{b}\right) - B_1\left(\frac{\beta(k_1)}{b}\right)\right][f(1) - f(b^{-1})]\right.\\ &\left.+ \frac{b^{-k_1+1}}{2}\left[B_2\left(\frac{\beta(k_1)+1}{b}\right) - B_2\left(\frac{\beta(k_1)}{b}\right)\right][f'(1) - f'(b^{-1})]\right]\\ &+ \frac{1}{2b}\,\frac{b^{-2k_2+2}}{2}\left[B_2\left(\frac{\beta(k_2)}{b}\right) + B_2\left(\frac{\beta(k_2)+1}{b}\right)\right]\\ &\times\left[\left[B_1\left(\frac{\beta(k_1)+1}{b}\right) - B_1\left(\frac{\beta(k_1)}{b}\right)\right][f'(1) - f'(b^{-1})]\right.\\ &\left.+ \frac{b^{-k_1+1}}{2}\left[B_2\left(\frac{\beta(k_1)+1}{b}\right) - B_2\left(\frac{\beta(k_1)}{b}\right)\right][f''(1) - f''(b^{-1})]\right]\\ &+ O(b^{-2k_1-k_2}),\end{aligned}$$

which reduces to the theorem. ■

Corollary 6.1: If $f \in C^2[b^{-1}, 1]$ and $k_2 > k_1$, then

$$p_f[\beta(k_1),\ \beta(k_2)] = b^{-2} + O(b^{-k_1}).$$

Theorem 7: If $f \in C^2[b^{-1}, 1]$ and $k_2 > k_1$, then

$$\operatorname{cov}_f(d_{k_1},\ d_{k_2}) = c_f b^{-k_1-k_2} + O(b^{-\min\{2k_1+k_2,\,2k_2\}}),$$

where

$$c_f = \left[\frac{(b-1)(b+1)}{12}\right]^2 [f'(1) - f'(b^{-1}) - (f(1) - f(b^{-1}))^2].$$

Proof: Write

$$\operatorname{cov}_f(d_{k_1}, d_{k_2}) = \sum_{\beta(k_1)=1}^{b-1} \sum_{\beta(k_2)=1}^{b-1} \beta(k_1)\beta(k_2)[p_f(\beta(k_1), \beta(k_2)) - p_f(\beta(k_1))p_f(\beta(k_2))].$$

Using the univariate and bivariate expressions of Theorems 3 and 6, respectively:

$$\operatorname{cov}_f(d_{k_1}, d_{k_2}) = b^{-k_1-k_2} \sum_{\beta(k_1)=1}^{b-1} \sum_{\beta(k_2)=1}^{b-1} \beta(k_1)\beta(k_2)\frac{1}{4}\left(\frac{2\beta(k_2)+1}{b} - 1\right) \times \left(\frac{2\beta(k_1)+1}{b} - 1\right)\Big[[f'(1) - f'(b^{-1})] - [f(1) - f(b^{-1})]^2\Big] + O(b^{-\min\{2k_1+k_2,\, 2k_2\}}).$$

Then, performing the two indicated sums yields the theorem. ■

Corollary 7.1: If $f \in C^2[b^{-1}, 1]$ and $k_2 > k_1$, then

$$\lim_{k_1+k_2\to\infty} \operatorname{cov}_f(d_{k_1}, d_{k_2}) = 0.$$

A second indicator of the weakening of dependence for large-digit numbers exists because it can be shown that the sequence $\{d_k\}$ of digit functions is *-mixing in the sense of Blum, Hanson, and Koopmans [13] when $f \in C^2[b^{-1}, 1]$ and $1/f$ is bounded above.

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