ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Proposed problems should be accompanied by their solutions. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-538 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that $\sqrt{5}g^n = gL_n + L_{n-1}$, where g is the golden ratio $(1 + \sqrt{5})/2$.

B-539 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $g = (1 + \sqrt{5})/2$ and show that

$$\left[1 + 2\sum_{i=1}^{\infty} g^{-3i}\right]\left[1 + 2\sum_{i=1}^{\infty} (-1)^i g^{-3i}\right] = 1.$$

B-540 *Proposed by A. B. Patel, V. S. Patel College of Arts & Sciences, Bilimora, India*

For $n = 2, 3, \ldots$, prove that

$$F_{n-1}F_nF_{n+1}L_{n-1}L_nL_{n+1}$$

is not a perfect square.

B-541 *Proposed by Heinz-Jürgen Seiffert, student, Berlin, Germany*

Show that $P_{n+3} + P_{n+1} + P_n \equiv 3(-1)^n L_n \pmod 9$, where the P_n are the Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and

$$P_{n+2} = 2P_{n+1} + P_n \quad \text{for } n \text{ in } N = \{0, 1, 2, \ldots\}.$$

B-542 *Proposed by Ioan Tomescu, University of Bucharest, Romania*

Find the sequence satisfying the recurrence relation

$$u(n) = 3u(n-1) - u(n-2) - 2u(n-3) + 1$$

and the initial conditions $u(0) = u(1) = u(2) = 0$.

B-543 *Proposed by P. Rubio, Dragados Y Construcciones, Madrid, Spain*

Let $a_0 = a_1 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for n in $Z^+ = \{1, 2, \ldots\}$. Find a simple formula for

$$G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k.$$

SOLUTIONS

Same Parity

B-514 *Proposed by Philip L. Mana, Albuquerque, N.M.*

Prove that $\binom{n}{5} + \binom{n+4}{5} \equiv n \pmod 2$ for $n = 5, 6, 7, \ldots$.

Solution by L. Cseh, student, Cluj, Romania

It is well known that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, for every $n > r$. Using this successively, we obtain:

$$\binom{n+4}{5} = \binom{n}{5} + 4\binom{n}{4} + 6\binom{n}{3} + 4\binom{n}{2} + \binom{n}{1}, \text{ for } n \geqslant 5.$$

From here:

$$\binom{n}{5} + \binom{n+4}{5} = 2\binom{n}{5} + 4\binom{n}{4} + 6\binom{n}{3} + 4\binom{n}{2} + n,$$

and so

$$\binom{n}{5} + \binom{n+4}{5} \equiv n \pmod 2 \quad \text{for } n = 5, 6, \ldots .$$

Also solved by Paul S. Bruckman, Adina Di Porto and Piero Filipponi, L. A. G. Dresel, C. Georghiou, Lawrence D. Gould, F. T. Howard, Walther Janous, M. S. Klamkin, H. Klauser, L. Kuipers, Graham Lord, Vania D. Mascioni, Imre Merenyi, George N. Philippou, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Smith, J. Suck, W. R. Utz, and the proposer.

Disguised Lucas Number

B-515 *Proposed by Walter Blumberg, Coral Springs, FL*

Let $Q_0 = 3$, and for $n \geqslant 0$, $Q_{n+1} = 2Q_n^2 + 2Q_n - 1$. Prove that $2Q_n + 1$ is a Lucas number.

Solution by C. Georghiou, University of Patras, Greece

We show that $2Q_n + 1 = L_2n + 2$. Let $R_n = 2Q_n + 1$. Then $R_0 = 7$, and for $n \geqslant 0$,

$$R_{n+1} = R_n^2 - 2. \qquad (*)$$

Now, using the identity $L_{4n} = L_{2n}^2 - 2$, it is easily verified that $R_n = L_{2n} + 2$ is a solution of (*). Since $R_0 = 7 = L_2 2$, $R_n = L_{2n} + 2$ is the unique solution of (*).

Also solved by Paul S. Bruckman, Laszlo Cseh, Adina Di Porto and Piero Filipponi, L. A. G. Dresel, Herta T. Freitag, Walther Janous, M. S. Klamkin, L. Kuipers, Graham Lord, Vania D. Mascioni, Imre Merenyi, George N. Philippou, Bob Prielipp, H.-J. Seiffert, A. G. Shannon, Sahib Singh, P. Smith, Lawrence Somer, J. Suck, M. Wachtel, Gregory Wulczyn, David Zeitlin, and the proposer.

Pell Equation Multiples of 36

B-516 *Proposed by Walter Blumberg, Coral Springs, FL*

Let U and V be positive integers such that $U^2 - 5V^2 = 1$. Prove that UV is divisible by 36.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria

From the theory of Pellian equations, it is very well known that starting from the minimal solution $u_0 = 9$, $v_0 = 4$, all solutions in natural numbers can be obtained via the recursion $u_{n+1} + v_{n+1}\sqrt{5} = (u_n + v_n\sqrt{5})(9 + 4\sqrt{5})$. Thus, the claim $36|UV$ can be shown by induction: $36|u_0 v_0 = 36$. Assume that $36|u_n v_n$. Since

$$u_{n+1}v_{n+1} = (9u_n + 20v_n)(4u_n + 9v_n) = 36(u_n^2 + 5v_n^2) + 161u_n v_n,$$

it follows at once that $36|u_{n+1}v_{n+1}$.

Also solved by Paul S. Bruckman, Laszlo Cseh, Adina Di Porto and Piero Filipponi, L. A. G. Dresel, C. Georghiou, Fuchin He, M. S. Klamkin, H. Klauser, Edwin M. Klein, L. Kuipers, Imre Merenyi, Bob Prielipp, H.-J. Seiffert, A. G. Shannon, Sahib Singh, P. Smith, Lawrence Somer, J. Suck, W. R. Utz, M. Wachtel, Gregory Wulczyn, and the proposer.

Square Sum of Adjacent Factorials

B-517 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Find all n such that $n! + (n + 1)! + (n + 2)!$ is the square of an integer.

Solution by Paul S. Bruckman, Fair Oaks, CA

Let $\theta_n = n! + (n + 1)! + (n + 2)!$; then

$$\theta_n = n!(1 + n + 1 + (n + 1)(n + 2)) = n!(n + 2)^2.$$

We see that θ_n is a square iff $n!$ is a square. Note that $\theta_0 = 1 + 1 + 2 = 2^2$ and $\theta_1 = 1 + 2 + 6 = 3^2$.

By Bertrand's Postulate, for any $n \geqslant 1$, there exists a prime p such that $n < p \leqslant 2n$. This, in turn, implies that for any $n \geqslant 2$, there exists a prime p such that $p \leqslant n < 2p$. Hence, if $n \geqslant 2$, $p|n!$ but $kp \nmid n!$ for all $k \geqslant 2$. In particular, $p^2 \nmid n!$. This shows that $n!$ cannot be a square if $n \geqslant 2$. Thus, the only values of n for which θ_n is square are $n = 0$ and $n = 1$.

Also solved by Laszlo Cseh, L. A. G. Dresel, Adina Di Porto and Piero Filipponi, C. Georghiou, Lawrence D. Gould, Fuchin He, Walther Janous, M. S. Klamkin, *Edwin M. Klein, L. Kuipers, Graham Lord, Vania D. Mascioni, Imre Merenyi, George N. Philippou, Bob Prielipp, Sahib Singh, Paul Smith, J. Suck, Gregory Wulczyn, H. Klauser, and the proposer.*

Fibonacci Inradius

B-518 *Proposed by Herta T. Freitag, Roanoke, VA*

Let the measures of the legs of a right triangle be

$$F_{n-1}F_{n+2} \quad \text{and} \quad 2F_nF_{n+1}.$$

What feature of the triangle has $F_{n-1}F_n$ as its measure?

Solution by L. A. G. Dresel, University of Reading, England

The sides of the right-angled triangle are given as

$$a = F_{n-1}F_{n+2} = (F_{n+1} - F_n)(F_{n+1} + F_n) = F_{n+1}^2 - F_n^2,$$
$$b = 2F_nF_{n+1};$$

hence,

$$a^2 + b^2 = (F_{n+1}^2 - F_n^2)^2 + 4F_n^2F_{n+1}^2 = (F_{n+1}^2 + F_n^2)^2$$

so that the third side is $c = F_{n+1}^2 + F_n^2$, and

$$a + b + c = 2F_{n+1}^2 + 2F_nF_{n+1} = 2F_{n+1}F_{n+2},$$

while $F_{n-1}F\ (a + b + c) = ab$ = twice the area of the triangle. It follows that $F_{n-1}F_n$ measures the radius r of the incircle, that is, the circle inscribed in the triangle and touching the three sides.

Also solved by Paul S. Bruckman, Laszlo Cseh, Adina Di Porto and Piero Filipponi, C. Georghiou, Lawrence D. Gould, Walther Janous, M. S. Klamkin, H. Klauser, L. Kuipers, Vania D. Mascioni, Imre Merenyi, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, Gregory Wulczyn, and the proposer.

Lucas Inradius

B-519 *Proposed by Herta T. Freitag, Roanoke, VA*

Do as in B-518 with each Fibonacci number replaced by the corresponding Lucas number.

Solution by L. A. G. Dresel, University of Reading, England

Since the proof for B-518 given above uses only the recurrence relation for the Fibonacci numbers $F_{n+1} = F_n + F_{n-1}$, etc., the corresponding result replacing each F_k by L_k can be proved in exactly the same way.

Also solved by the solvers of B-518 and the proposer.

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