A LUCAS TRIANGLE PRIMALITY CRITERION DUAL TO THAT OF MANN-SHANKS

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Consider the following array of numbers

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2										
2	1	3	2									
3	1	4	5	2								
4	1	5	9	7	2							
5	1	6	14	16	9	2						
6	1	7	20	30	25	11	2					
7	1	8	27	50	55	36	13	2				
8	1	9	35	77	105	91	49	15	2			
9	1	10	44	112	182	196	140	64	17	2		
10	1	11	54	156	294	378	336	204	81	19	2	
11	1	12	65	210	450	672	714	540	285	100	21	2

where any element in the array is found by the usual Pascal recurrence, i.e.,

$$A(n, k) = A(n - 1, k) + A(n - 1, k - 1), \tag{1}$$

subject to the initial conditions $A(1, 0) = 1$, $A(1, 1) = 2$, with $A(n, k) = 0$ for $k < 0$ or $k > n$. This array has been called a Lucas triangle by Feinberg [1], because rising diagonals sum to give the Lucas numbers 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, ..., in contrast to the rising diagonals in the standard Pascal triangle where rising diagonals sum to give the Fibonacci numbers 1, 1, 2, 3, 5, 8, ... The seventh diagonal in our array is 1, 7, 14, 7; the eleventh diagonal is 1, 11, 44, 77, 55, 11. This suggests the following.

Theorem 1. The number $D \geqslant 2$ is a prime number if and only if every entry that is greater than 1 along the D^{th} rising diagonal in the Lucas triangle is divisible by D.

Before giving a proof, we set down further notation in order to rephrase the theorem.

It is easy to prove directly from (1), or one can quote the general theorem of Gupta [4], that

$$A(n, k) = \binom{n}{k} + \binom{n - 1}{k - 1}, \tag{2}$$

so that the Lucas triangle is simply a combination of two shifted Pascal triangles. Let D be the diagonal number in question and let j be the position of an entry along that diagonal, then a typical element of the diagonal is given by $A(D - j, j)$, where $0 \leqslant j \leqslant D/2$. We can now rephrase Theorem 1 as follows.

Theorem 2. $D \geqslant 2$ is a prime number if and only if $D|A(D-j, j)$ for all j such that $1 \leqslant j \leqslant D/2$.

Proof: We have from (2) that

$$A(D - j, j) = \binom{D-j}{j} + \binom{D-j-1}{j-1} = D\binom{D-j-1}{D-2j}\Big/j$$
$$= D(D - j - 1)!/j!(D - 2j)!.$$

If $D = p$ is a prime $\geqslant 2$, we observe that $(j!, p) = 1$ and $((p - 2j)!, p) = 1$ for $1 \leqslant j \leqslant p/2$ so that surely $j!(p - 2j)!\,|\,(p - j - 1)!$ and therefore p is a factor of the number $p \cdot ((p - j - 1)!/j!(p - 2j)!)$.

Now suppose that D is composite. Then, from the formula for A,

$$D|A(D - j, j) \text{ if and only if } D\Big|D\binom{D-j-1}{D-2j}\Big/j.$$

We will show that for a composite D, some j cannot divide $\binom{D-j-1}{D-2j}$. Recall that for the binomial coefficients we have $\binom{x}{m} = (-1)^m\binom{-x+m-1}{m}$. Therefore

$$j\,\Big|\binom{D-j-1}{D-2j} \text{ if and only if } j\,\Big|\binom{-j}{D-2j}.$$

we need not consider the question of divisilibity of the entries in any diagonal by D when D is even, since the last entry is always a 2 for even D, so we can restrict our analysis to odd composite $D > 3$. Put $D = p(2k + 1)$, where p is an odd prime factor of D, and choose $j = pk$. Then we are concerned with whether

$$pk\,\Big|\binom{-pk}{p}.$$

But

$$\frac{1}{pk}\binom{-pk}{p} = \frac{-(-pk-1)(-pk-2)\cdots(-pk-p+1)}{p(p-1)(p-2)\cdots 3\cdot 2\cdot 1}, \tag{3}$$

and we observe that the factors $p-1, p-2, \ldots 3, 2$ cannot affect the divisibility of the numerator by p since $(p, p - r) = 1$ for all $1 \leqslant r \leqslant p - 1$. Furthermore, p is relatively prime to every factor in the numerator; that is,

$$(p, pk + s) = 1 \quad \text{for all } 1 \leqslant s \leqslant p - 1,$$

and so the indicated quotient cannot be an integer. This completes the proof.

We now claim that Theorem 2 is a dual to the criterion discovered by Mann and Shanks [7]. In [2] and [3] it is shown that the Mann-Shanks criterion can be restated as follows.

Theorem 3. The number $C \geqslant 2$ is a prime number if and only if

$$R\,\Big|\binom{R}{C-2R} \tag{4}$$

for all $R \geqslant 1$ such that $C/3 \leqslant R \leqslant C/2$.

Comparison of our proof of the new prime criterion with that of the Mann-Shanks criterion in [2], [3], and [7] shows that the same considerations have been made using (3), except that the numerator in the earlier proof was

$$(pk - 1)(pk - 2)\cdots(pk - p + 1)$$

and the minus sign made no difference in the argument. In fact, we see that our new criterion may be restated as follows.

Theorem 4. The number $C \geqslant 2$ is a prime number if and only if

$$R \left| \binom{-R}{C - 2R} \right. \tag{5}$$

for all R such that $1 \leqslant R \leqslant C/2$.

The natural display for our criterion is the Lucas triangle, just as the natural display for the Mann-Shanks criterion is their shifted Pascal triangle.

Since the rising diagonals in the Lucas triangle sum to give Lucas numbers, that is, as Feinberg [1] noted,

$$L_n = \sum_{j=0}^{[n/2]} A(n - j, j) = 1 + \sum_{j=1}^{[n/2]} \frac{n}{j}\binom{n - j - 1}{j - 1} \tag{6}$$

where $L_n = \alpha^n + \beta^n$ with α and β the roots of the equation $x^2 - x - 1 = 0$, and $L_{n+1} = L_n + L_{n-1}$, subject to $L_1 = 1$, $L_2 = 3$, then we have an obvious

Corollary. The Lucas numbers satisfy the congruence

$$L_p \equiv 1 \pmod{p} \tag{7}$$

for all primes $p \geqslant 2$.

This corollary is well known and can be found in Lehmer [6] or in [8].

That the converse of (7) does not hold follows from the well-known counterexample of Hoggatt and Bicknell that

$$L_{705} \equiv 1 \pmod{705 = 3 \cdot 5 \cdot 47},$$

although Lind [8] used computer calculations to show that, for all $2 \leqslant n < 700$, $L_n \equiv 1 \pmod{n}$ implies that n is prime.

In later papers, we shall exhibit and prove corresponding duals to the extensions of the Mann-Shanks criterion given in [2] and [3].

Remark: It is interesting to compare the criterion discussed here with the familiar fact that

$$n \left| \binom{n}{k} \right. \text{ for all with } 1 \leqslant k < n \text{ if and only if } n \text{ is a prime.}$$

Harborth [5] has shown that "almost all" binomial coefficients $\binom{n}{k}$ are divisible by their row number n.

Finally, we note that the generating function for the A's is clearly

$$(1 + 2x)(1 + x)^{n-1} = \sum_{k=0}^{n} A(n, k)x^k. \tag{8}$$

The results of this paper were first announced in an abstract [9] in 1977. There is now a rather extensive international bibliography on criteria related to the Mann-Shanks theorem, and we hope to summarize this at a later date.

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