ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-394 *Proposed by Ambati Jaya Krishna, Baltimore, MD, and Gomathi S. Rao, Orangeburg, SC*

Find the value of the continued fraction $1 + \frac{2}{3 +} \frac{4}{5 +} \frac{6}{7 +} \cdots$.

H-395 *Proposed by Heinz Jürgen Seiffert, Berlin, Germany*

Show that for all positive integers m and k,

$$\sum_{n=0}^{m-1} \frac{F_{2k(2n+1)}}{L_{2n+1}} = \sum_{j=0}^{k-1} \frac{F_{2m(2j+1)}}{L_{2j+1}}$$

H-396 *Proposed by M. Wachtel, Zürich, Switzerland*

Establish the identity:

$$\sum_{i=1}^{\infty} \frac{F_{i+n}}{a^i} + \sum_{i=1}^{\infty} \frac{F_{i+n+1}}{a^i} = \sum_{i=1}^{\infty} \frac{F_{i+n+2}}{a^i}$$

$a = 2, 3, 4, \ldots, n = 0, 1, 2, 3, \ldots$.

A reply regarding H-354 by M. Wachtel, Zürich, Switzerland

In a note in the May 1985 issue, the proposer is claiming that my solution which appeared in the August 1984 issue is not a solution.

Reply: After having unsuccessfully attempted to understand the argumentation given in the above note, I might restrict myself to the following:

1. Admittedly, the theories I developed are not a solution in a strict mathematical sense; neither was it intended to evoke this impression, since no proofs were given. I believe, however, that these theories are new ones, and as shown, they lead to the desired solutions of the equation ($Ax^2 + C = By^2$) in integers.

2. The proposer claims that I attempted construction of the solutions to particular cases. Yes, but I designated them clearly as arbitrary examples, and, moreover, I also mentioned in §1.2: "Considering the limited space, only main fragments of the whole issue can be dealt with here." Not one, but many formulas are involved, I surmise.

3. The problems H-350 and H-372, proposed by myself and mentioned by Bruckman, are particular instances of the equation ($Ax^2 + C = By^2$) and solvable by the theories I described.

4. Bruckman states: "Moreover, an explicit formula for all such solutions is known, in terms of the one known solution." Frankly, I cannot imagine that there exists a general formula which would cope with particular values of C, or with the sometimes amazing complexity of the relations of A to B. In §2.3, I outlined: "To determine (x_2, y_2), there does not (presumably) exist a general formula, but an undeterminable number of different construction rules, according to the group or class to which the sequence belongs. When both (x_1, y_1) and (x_2, y_2) are found, all other terms are determined." I have found quite a lot of such construction rules to determine (x_2, y_2).

As to the "explicit formula" for all such solutions, I wonder if, e.g., for ($A = 11(L_5)$, $x_1 = 2$, $C = 3$, $B = 47(L_8)$, $y_1 = 1$) the desired sequence can be established.

5. As an autodidact in mathematics with no high school education, I am, naturally, sometimes unable to observe a strong mathematical way. In conclusion, may I observe that Bruckman in this note quoted my name incorrectly.

SOLUTIONS

Primitive Sequences

H-369 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC (Vol. 22, no. 2, May 1984)*

Call an integer-valued arithmetic function f a *gcd sequence* if

$$\gcd(a, b) = d \quad \text{implies} \quad \gcd(f(a), f(b)) = f(d)$$

for all positive integers a and b. A gcd sequence is *primitive* if it is neither an integer multiple nor a positive integer power of some other gcd sequence. Examples of primitive gcd sequences include:

(1) $f(n) = 1$
(2) $f(n) = n$
(3) $f(n)$ = largest squarefree divisor of n
(4) $f(n) = 2^n - 1$
(5) $f(n) = F_n$ (Fibonacci sequence)

Prove that there are infinitely many primitive gcd sequences.

Solution by Paul S. Bruckman, Fair Oaks, CA

Let G and PG represent the sets of gcd sequences and primitive gcd sequences, respectively. There is a possible misstatement in the definition of of gcd sequence given in the statement of the problem, which requires f to be an *arithmetic* function; recall a function $f: N \to N$ is *arithmetic* if $f(1) = 1$ and $f(mn) = f(m) \quad f(n)$ whenever $\gcd(m, n) = 1$. However, $f_4(n) \equiv 2^n - 1$ and

$f_5(n) \equiv F_n$ are *not* arithmetic functions, even though these functions have the "gcd property" stated in the problem. Assuming that the proposer did not intend the offending word *arithmetic* in the definition of G and its subset PG, no difficulty arises.

Infinitely many sequences $(f_n) \in G$ are then generated by the recursion:

$$(1) \qquad f_{n+2}(x) = xf_{n+1}(x) + f_n(x),\ n = 0, 1, 2, \ldots;\ f_0(x) = 0,\ f_1(x) = 1,$$

where x is any positive integer.

The $f_n(x)$'s given above are generalized Fibonacci polynomials. It was shown by Hoggatt and Long ["Divisibility Properties of Generalized Fibonacci Polynomials," *The Fibonacci Quarterly* **12**, no. 2 (1974):113-130] that these polynomials have the gcd property, that is,

$$(2) \qquad \gcd(f_m(x),\ f_n(x)) = f_{\gcd(m,n)}(x).$$

Hence $(f_n(x)) \in G$. Since $f_1(x) = 1$, $f_n(x)$ is not a multiple of another sequence in G. Also, we may choose $x = f_2(x)$ to be a non-power, in infinitely many ways; with such choices, we see that $(f_n(x))$ cannot be a power of another sequence in G. Hence $(f_n(x))$ PG for infinitely many choices for x. Q.E.D.

Also solved by W. Janous, L. Kuipers, L. Somer, and the proposer.

Lotsa Fives in the Product

H-370 *Proposed by M. Wachtel and H. Schmutz, Zürich, Switzerland (Vol. 22, no. 2, May 1984)*

For every positive integer a show that

(A) $5 \cdot [5 \cdot (a^2 + a) + 1] + 1$

(B) $5 \cdot [5 \cdot [5 \cdot [5 \cdot (a^2 + a) + 1] + 1] + 1] + 1$

are *products of two consecutive integers*, and that no integral divisor of

$$5 \cdot (a^2 + a) + 1$$

is congruent to 3 or 7, modulo 10.

Solution by Lawrence Somer, Washington, D.C.

Expanding expression (A), we obtain

$$25a^2 + 25a + 6,$$

which is the product of the consecutive integers $5a + 2$ and $5a + 3$. Expanding expression (b), we obtain

$$625a^2 + 125a + 156,$$

which is the product of the consecutive integers $25a + 12$ and $25a + 13$.

In general, it can be shown by induction that if

$$S_k(a) = \underbrace{5 \cdot [5 \cdot [5 \cdot \cdots \cdot [5}_{2k\ 5\text{'s}} \cdot (a^2 + a) + \underbrace{1] + 1] + \cdots + 1] + 1}_{2k\ 1\text{'s}},$$

where k is a fixed positive integer, then

$$S_k(a) = (5^k a + (5^k - 1)/2) \cdot (5^k a + ((5^k - 1)/2) + 1).$$

By looking at the ring of integers modulo 10, one sees that an integer is congruent to 3 or 7 modulo 10 if and only if at least one of its prime divisors is congruent to 3 or 7 modulo 10. Thus, to prove the last part of the problem, we need only show that no prime divisor $5(a + a) + 1$ is congruent to 3 or 7 modulo 10. Let p be a prime such that $p = 3$ or 7 modulo 10. Then, by the law of quadratic reciprocity, $(5/p) = -1$, where $(5/p)$ is the Legendre symbol. Suppose

$$5(a^2 + a) + 1 \equiv 0 \pmod{p}.$$

Multiplying both sides by 20 and then adding 5 to both sides, one obtains

$$(100a^2 + 100a + 25) = (10a + 5)^2 \equiv 5 \pmod{p}.$$

However, this is a contradiciton, since $(5/p) = -1$. We are now done.

Also solved by P. Bruckman, O. Brugia & P. Filipponi, L. Dresel, F. He, J. Metzger, B. Prielipp, and the proposers.

Continuing ...

H-371 *Proposed by Paul S. Bruckman, Fair Oaks, CA (Vol. 22, no. 2, May 1984)*

Let $[\overline{k}]$ represent the purely periodic continued fraction:

$$k + 1/(k + 1/(k + \cdots, k = 1, 2, 3, \ldots .$$

Show that

$$[\overline{k}]^3 = [\overline{k^3 + 3k}].$$

Generalize to other powers.

Solution by O. Brugia, A. Di Porto, & P. Filipponi, Fdn. U. Bordoni, Rome, Italy

Let δ_m be the m^{th} convergent of $[\overline{k}]$; as known [1], both the numerator P_m and the denominator Q_m of δ_m can be expressed by the same difference equation, $R_m = kR_{m-1} + R_{m-2}$, with initial conditions $R_0 = 1$, $R_1 = k$ for P_m, and $R_0 = 0$, $R_1 = 1$ for Q_m. Since the roots of the corresponding characteristic equation $z^2 - kz - 1 = 0$ are $z_1 = (k - \sqrt{k^2 + 4})/2$ and $z_2 = (k + \sqrt{k^2 + 4})/2$, we get $\delta_m = (z_2^{m+1} + z_1^{m+1})/(z_2^m - z_1^m)$, and hence

$$[\overline{k}] = \lim_{m \to \infty} \delta_m = z_2 = (k + \sqrt{k^2 + 4})/2, \text{ for } k > 0. \quad (1)$$

For every nonnegative integer n we will find, if any, a nonnegative integer h_n such that

$$[\overline{h}_n] = [\overline{k}]^n. \quad (2)$$

From (1), equation (2) can be rewritten as $(h_n + \sqrt{h_n^2 + 4})/2 = z_2^n$ and gives

$$h_n = (z_2^{2n} - 1)/z_2^n = z_2^n - (-z_1)^n$$
$$= ((\sqrt{k^2 + 4} + k)/2)^n - ((\sqrt{k^2 + 4} - k)/2)^n, \quad (3)$$

where use has been made of the relation

$$z_1 z_2 = -1. \quad (4)$$

Because

$$h_n = \begin{cases} 0, & \text{for } n = 0 \\ \dfrac{2}{2^{2\nu}}\sqrt{k^2+4}\displaystyle\sum_{i=0}^{\nu-1}\binom{2\nu}{2i+1}(k^2+4)^i k^{2\nu-2i-1}, & \text{for } n = 2\nu,\ \nu = 1, 2, \ldots \\ \dfrac{1}{2^{2\nu}}\displaystyle\sum_{i=0}^{\nu}\sum_{j=0}^{i}\binom{2\nu+1}{2i}\binom{i}{j}2^{2j}k^{2\nu-2j+1}, & \text{for } n = 2\nu+1,\ \nu = 0, 1, \ldots \end{cases} \tag{5}$$

we see that $h_{2\nu}$ is irrational for $\nu \neq 0$ and $h_{2\nu+1}$ is rational. Moreover, since (5) becomes

$$h_{2\nu+1} = \sum_{\mu=0}^{\nu} \lambda_{2\nu+1,2\mu+1} k^{2\mu+1} \tag{6}$$

where

$$\lambda_{2\nu+1,2\mu+1} = \frac{1}{2^{2\mu}}\sum_{i=\nu-\mu}^{\nu}\binom{2\nu+1}{2i}\binom{i}{\nu-\mu}$$

$$= \frac{1}{2^{2\mu}}\sum_{i=0}^{\mu}\binom{2\nu+1}{2\mu-2i+1}\binom{i+\nu-\mu}{i}, \tag{7}$$

it can be shown that $h_{2\nu+1}$ is a positive integer.

First of all, we observe that the right-hand side of (7) is a polynomial in ν having:

- degree $2\mu + 1$,
- the coefficient of $\nu^{2\mu+1}$ equal to $2/(2\mu + 1)!$
- the first $\mu + 1$ roots equal to $-\frac{1}{2}$ and $\nu_r = r$ $(r = 0, 1, \ldots, \mu - 1)$ because either $\binom{2\nu+1}{2\mu-2i+1}$ or $\binom{i+\nu-\mu}{i}$ vanishes for these values of ν and $0 \leqslant i \leqslant \mu$.

To find the remaining μ roots of the above polynomial, we utilize the identity

$$\lambda_{-(2\nu+1),2\mu+1} = -\lambda_{2\nu+1,2\mu+1} \tag{8}$$

derived from (3), (4), and (6). Setting $\nu = -\nu_r - 1$ into (7) and using (8), we have

$$\lambda_{2(-\nu_r-1)+1,2\mu+1} = \lambda_{-(2\nu_r+1),2\mu+1} = -\lambda_{2\nu_r+1,2\mu+1} = 0$$

because the ν_r's are roots of (7), and therefore also $-\nu_r - 1 = -r - 1$ $(r = 0, 1, \ldots, \mu - 1)$ are roots of (7).

On the basis of the previous observations, we have the result:

$$\lambda_{2\nu+1,2\mu+1} = \frac{2}{(2\mu+1)!}\left(\nu+\frac{1}{2}\right)\prod_{r=0}^{\mu-1}(\nu-r)(\nu+r+1)$$

$$= \frac{2\nu+1}{2\mu+1}\binom{\nu+\mu}{2\mu} = 2\binom{\nu+\mu}{2\mu+1} + \binom{\nu+\mu}{2\mu}. \tag{9}$$

Since (9) shows that the $\lambda_{2\nu+1,2\mu+1}$'s are positive integers, we conclude that the $h_{2\nu+1}$'s are positive integers as well. The values of $\lambda_{2\nu+1,2\mu+1}$, for $0 \leqslant \nu \leqslant 4$, are given in the table at the right.

ν \ μ	0	1	2	3	4
0	1				
1	3	1			
2	5	5	1		
3	7	14	7	1	
4	9	30	27	9	1

The second row of the table, together with (6) and (2) shows that

$$[\overline{k}]^3 = [\overline{3k + k^3}].$$

Reference: [1] I. M. Vinogradov. *Elements of Number Theory*. New York: Dover Publications, 1954.

Also solved by P. Bruckman, F. He, W. Janous, L. Kuipers, M. Wachtel, and the proposer.

Recurring Thoughts

H-372 *Proposed by M. Wachtel, Zurich, Switzerland*
(Vol. 22, no. 3, August 1984)

There exist infinitely many sequences, each with infinitely many solutions of the form:

$\underline{A}\cdot x_1^2 + C = \underline{B}\cdot y_1^2$	$\underline{A} = F_{n+3}$	$\underline{C} = L_n$	$\underline{B} = F_{n+1}$
$\underline{A}\cdot x_2^2 + C = \underline{B}\cdot y_2^2$	$\underline{x_1} = 1$		$\underline{y_1} = 2$
$\underline{A}\cdot x_3^2 + C = \underline{B}\cdot y_3^2$	$\underline{x_2} = F_{n-1}F_n + F_{n+1}^2$		$\underline{y_2} = 2F_{n+1}^2$
....			
$\underline{A}\cdot x_m^2 + C = \underline{B}\cdot y_m^2$	$\underline{x_3} = 2F_{2n+4} + (-1)^n$		$\underline{y_3} = 2F_{2n+5}$

Find a recurrence formula for x_4/y_4, x_5/y_5, ..., x_m/y_m (y_m = dependent on x_m).

Examples: $(x_1 - x_3)$

	(in numbers)
$\underline{n = 3}$	
$\underline{F_6}\cdot(1)^2 + \underline{L_3} = \underline{F_4}\cdot(2)^2$	$8\cdot 1 + \underline{4} = \underline{3}\cdot 2^2$
$\underline{F_6}\cdot(F_2F_3+F_4^2)^2 + \underline{L_3} = \underline{F_4}\cdot(2F_4^2)^2$	$\underline{8}\cdot 11^2 + \underline{4} = \underline{3}\cdot 18^2$
$\underline{F_6}\cdot(2F_{10} - 1)^2 + \underline{L_3} = \underline{F_4}\cdot(2F_{11})^2$	$\underline{8}\cdot 109^2 + \underline{4} = \underline{3}\cdot 178^2$
$\underline{n = 4}$	
$\underline{F_7}\cdot(1)^2 + \underline{L_4} = \underline{F_5}\cdot(2)^2$	$\underline{13}\cdot 1 + \underline{7} = \underline{5}\cdot 2^2$
$\underline{F_7}\cdot(F_3F_4 + F_5^2)^2 + \underline{L_4} = \underline{F_5}\cdot(2F_5^2)^2$	$\underline{13}\cdot 31^2 + \underline{7} = \underline{5}\cdot 50^2$
$\underline{F_7}\cdot(2F_{12} + 1)^2 + \underline{L_4} = \underline{F_5}\cdot(2F_{13})^2$	$\underline{13}\cdot 289^2 + \underline{7} = \underline{5}\cdot 466^2$

Solution by Paul S. Bruckman, Fair Oaks, CA

Let

$$r_m = x_m/y_m,\quad m \in \mathbb{Z}, \tag{1}$$

where (x_m, y_m) are the solutions (if any) of the equation:

$$F_{n+1}y^2 - F_{n+3}x^2 = L_n, \tag{2}$$

and n is a fixed nonnegative integer. This is a particular instance of the general equation:

$$ay^2 - bx^2 = c, \tag{3}$$

where a, b, and c are pairwise relatively prime positive integers, with a and b not both perfect squares.

By a *solution* of (3), we mean any ordered pair of integers (x, y) solving (3), but with $y > 0$. This allows trivial sign variations in the x-coordinate but not in the y-coordinate; the theory is much more elegant with this convention. We then write $(x, y) \in \mathcal{S}(a, b, c)$ iff (x, y) is a solution of (3).

We may infer from the theory of such equations that the solution set $\mathcal{S}(a, b, c)$ (if nonempty) is generated from the set $\mathcal{S}(1, ab, 1)$. More specifically, if $(x_m, y_m) \in \mathcal{S}(a, b, c)$ and $(p_m, q_m) \in \mathcal{S}(1, ab, 1)$, then

$$x_m = ay_0 p_m + x_0 q_m, \qquad (4)$$
$$y_m = bx_0 p_m + y_0 q_m, \quad m \in \mathbb{Z}, \text{ where } (x_0, y_0) \text{ is any solution of (3).}$$

It is an easy exercise to show that the expressions given by (4) do, in fact, provide solutions of (3), given that $q_m^2 - abp_m^2 = 1$.

For our most specific case, we first show that

$$(1, 2) \in \mathcal{S}(F_{n+1}, F_{n+3}, L_n).$$

For $4F_{n+1} - F_{n+3} = 3F_{n+1} - F_{n+2} = 2F_{n+1} - F_n = F_{n+1} + F_{n-1} = L_n$. Thus, (1, 2) is a solution of (2); clearly, it is the minimal solution. We will find it convenient to choose $(x_0, y_0) = (1, 2)$. We then need to solve the auxiliary equation:

$$v^2 - F_{n+1}F_{n+3}u^2 = 1, \qquad (5)$$

then substitute in (4) to obtain all solutions (x_m, y_m) of (2), with

$$x_0 = 1, \; y_0 = 2, \; a = F_{n+1}, \; b = F_{n+3}.$$

Note that $F_{n+1}F_{n+3} = F_{n+2}^2 + (-1)^n$. The general solutions of (5) are generated by expansion of powers of the quantities α and β defined by:

$$\alpha \equiv F_{n+2} + \sqrt{F_{n+1}F_{n+3}}, \quad \beta \equiv F_{n+2} - \sqrt{F_{n+1}F_{n+3}}. \qquad (6)$$

Note that

$$\alpha\beta = (-1)^{n+1}, \; \alpha + \beta = 2F_{n+2}, \; \alpha - \beta = 2\sqrt{F_{n+1}F_{n+3}}.$$

If we make the following definitions:

$$u_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}, \quad v_m = \tfrac{1}{2}(\alpha^m + \beta^m), \; m \in \mathbb{Z}, \qquad (7)$$

we see that $v_m^2 - F_{n+1}F_{n+3}u_m^2 = \frac{1}{4}(\alpha^m + \beta^m)^2 - \frac{1}{4}(\alpha^m - \beta^m)^2 = \frac{1}{2} \cdot 4(\alpha\beta)^m$, or

$$v_m^2 - F_{n+1}F_{n+3}u_m^2 = (-1)^{m(n+1)}. \qquad (8)$$

From the definitions in (7), we may derive the following identities, which are indicated without proof:

$$u_{m+1}v_m - u_m v_{m+1} = (-1)^{(n+1)m}; \qquad (9)$$

$$v_m v_{m+1} - F_{n+1}F_{n+3}u_m u_{m+1} = (-1)^{(n+1)m}F_{n+2}; \qquad (10)$$

$$u_{m+2}v_m - u_m v_{m+2} = (-1)^{(n+1)m}2F_{n+2}; \qquad (11)$$

$$v_m v_{m+2} - F_{n+1}F_{n+3}u_m u_{m+2} = (-1)^{(n+1)m}(2F_{n+2}^2 - 1). \qquad (12)$$

We see from (8) that if n is odd, $(u_m, v_m) \in \mathcal{S}(1, F_{n+1}F_{n+3}, 1)$. Setting $a = F_{n+1}$, $b = F_{n+3}$, $x_0 = 1$, $y_0 = 2$, $p_m = u_m$, $q_m = v_m$ in (4), we thus obtain the explicit solutions of (1), if n is odd:

$$x_m = 2F_{n+1}u_m + v_m, \; y_m = F_{n+3}u_m + 2v_m, \; m \in \mathbb{Z}. \qquad (13)$$

If n is even, we see from (8) that $(u_m, v_m) \in \mathcal{S}(1, F_{n+1}F_{n+3}, 1)$ iff m is even. Hence, we make the same substitutions in (4) as for the case where n is odd, except that now we set $p_m = u_{2m}$, $q_m = v_{2m}$. Thus, the general solutions of (1) if n is even are as follows:

$$x_m = 2F_{n+1}u_{2m} + v_{2m}, \; y_m = F_{n+3}u_{2m} + 2v_{2m}, \; m \in \mathbb{Z}. \qquad (14)$$

Next, we derive a pair of useful relations involving successive values of (x_m, y_m):

$$x_{m+1}y_m - x_m y_{m+1} = \begin{cases} L_n, & n \text{ odd}, \\ 2L_nF_{n+2}, & n \text{ even}; \end{cases} \tag{15}$$

$$F_{n+1}y_m y_{m+1} - F_{n+3}x_m x_{m+1} = \begin{cases} L_nF_{n+2}, & n \text{ odd}, \\ L_n(2F_{n+2} + 1), & n \text{ even}. \end{cases} \tag{16}$$

For brevity, we write $u = u_m$ or u_{2m}, $u' = u_{m+1}$ or u_{2m+2}, depending on whether n is odd or even, respectively, with similar notation for v and v'.

Substituting the expressions in (13) or (14) into (15) and using (9) or (11), as appropriate, the left member of (15) becomes, after simplification:

$$(u'v - uv')(4F_{n+1} - F_{n+3}) = \begin{cases} (-1)^{(n+1)m}L_n = L_n, & n \text{ odd}; \\ (-1)^{(n+1)2m}2F_{n+2}L_n = 2L_nF_{n+2}, & n \text{ even}. \end{cases}$$

Likewise, substituting the expressions in (13) or (14) into (16) and using (10) or (12), as appropriate, the left member of (16) becomes, after simplification:

$$(vv' - F_{n+1}F_{n+3}uu')(4F_{n+1} - F_{n+3}) = \begin{cases} (-1)^{(n+1)m}F_{n+2}L_n = L_nF_{n+2}, & n \text{ odd}; \\ (-1)^{(n+1)2m}(2F_{n+2}^2 - 1)L_n = L_n(2F_{n+2}^2 - 1), & n \text{ even}. \end{cases}$$

This completes the proof of (15) and (16).

Using (15) and (16), we may now derive the desired recursion for the r_m's. Dividing (15) and (16) throughout by $y_m y_{m+1}$, we obtain:

$$r_{m+1} - r_m = A/y_m y_{m+1}, \quad F_{n+1} - F_{n+3}r_{m+1}r_m = B/y_m y_{m+1},$$

where

$$A = \begin{cases} L_n, & n \text{ odd}; \\ 2L_nF_{n+2}, & n \text{ even}; \end{cases} \qquad B = \begin{cases} L_nF_{n+2}, & n \text{ odd}; \\ L_n(2F_{n+2}^2 + 1), & n \text{ even}. \end{cases}$$

Thus,

$$F_{n+1} - F_{n+3}r_{m+1}r_m = (B/A)(r_{m+1} - r_m).$$

Solving for r_{m+1}, we find:

$$r_{m+1} = \frac{Br_m + AF_{n+1}}{AF_{n+3}r_m + B}. \tag{17}$$

In terms of the functions of n, we find that each of the terms in the fraction in (17) contains the constant term L_n, which may be cancelled. Hence, in simplest terms, we obtain the two expressions:

$$r_{m+1} = \begin{cases} \dfrac{F_{n+2}r_m + F_{n+1}}{F_{n+3}r_m + F_{n+2}}, & n \text{ odd}; \\[2ex] \dfrac{(2F_{n+2}^2 + 1)r_m + 2F_{n+1}F_{n+2}}{2F_{n+2}F_{n+3}r_m + 2F_{n+2}^2 + 1}, & n \text{ even}. \end{cases} \tag{18}$$

We may also solve for r_m in terms of r_{m+1}, thus obtaining:

$$r_m = \frac{F_{n+2}r_{m+1} - F_{n+1}}{-F_{n+3}r_{m+1} + F_{n+2}}, \quad n \text{ odd};$$
$$r_m = \frac{(2F_{n+2}^2 + 1)r_{m+1} - 2F_{n+1}F_{n+2}}{-2F_{n+2}F_{n+3}r_m + 2F_{n+2}^2 + 1}, \quad n \text{ even}. \tag{19}$$

Using (18) and (19), we may extend the sequence

$$(r_m)_{-\infty}^{\infty}$$

in either direction, given $r_0 = \frac{1}{2}$.

Also solved by the proposer

◆◇◆◇◆