RAPIDLY CONVERGING EXPANSIONS WITH FIBONACCI COEFFICIENTS

DARIO CASTELLANOS
Universidad de Carabobo, Valencia, Venezuela
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1. FIBONACCI NUMBERS AND CHEBYCHEV POLYNOMIALS

Properties of Fibonacci numbers have been known for a very long time. Their origin dates back to the year 1202 with the publication of the *Liber Abaci* by the Italian mathematician Leonardo of Pisa, better known to us by the nickname "Fibonacci," a short form of *Filius Bonacci*, meaning "Son of Bonacci."

Fibonacci seems to have had a sense of humor apart from his mathematical talents: Liber was a Latin God, son of Ceres and brother of Proserpina. The Romans assimilated this God to Bacchus or Dyonisus, the Greek god of wine. Festivals, known as "Liberalia," were celebrated every year honoring *Liber Bacus*. Since *Liber Abaci* means *Book of the Abacus*, Fibonacci may have amused himself by naming his book, at a time of strong domination by the Roman Catholic Church, in a way reminiscent of a pagan god of wine and fertility. We know Fibonacci was fond of play on words. For instance, he signed some of his work "Leonardo Bigollo." *Bigollo* is a work meaning both "traveler," which Fibonacci certainly was, and "blockhead." It has been said that Fibonacci had in mind the latter meaning to tease his contemporaries who had ridiculed him for his interest in Hindu-Arabic numerals and methods. Fibonacci had become a very successful mathematician whith these methods.

Fibonacci did not discover any of the properties of the sequence which bears his name. He simply proposed, and solved, in the *Liber Abaci*, the problem of how many rabbits would be born in one year starting from a given pair. With some natural assumptions about the breeding habits of rabbits, the population of rabbit pairs per month correspond to the elements of the Fibonacci sequence—1, 1, 2, 3, 4, 8, 13, etc.—where, beginning with zero and one, each term of the sequence is the sum of the two preceding ones.

With the passage of time, this sequence would appear in so many areas with no possible connection to the breeding of rabbits that, in 1877, Edward Lucas proposed naming it *Fibonacci Sequence* and its terms *Fibonacci Numbers*. The fertility of this sequence seems to be inexhaustible, and every year new and curious properties of it are discovered. F_n has become the standard symbol for Fibonacci numbers, and their defining relation is

$$F_n = F_{n-1} + F_{n-2},\ F_0 = 0,\ F_1 = 1.$$

In spite of the above preamble, it will, perhaps, appear as surprising to encounter some new, simple, and unexpected relations between Fibonacci numbers and Chebychev polynomials. Let us proceed to their derivation.

The known relation [8] for Chebychev polynomials of the first kind,

$$T_n(x) = \tfrac{1}{2}[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n], \tag{1}$$

gives, with $x = \sqrt{5}/2$,

$$\frac{2}{\sqrt{5}} T_n\left(\frac{\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^n + (-1)^n\left(\frac{1 - \sqrt{5}}{2}\right)^n\right]. \tag{2}$$

For odd n, this relation coincides with Binet's formula [6] for Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{3}$$

Thus, we obtain

$$F_{2n+1} = \frac{2}{\sqrt{5}} T_{2n+1}\left(\frac{\sqrt{5}}{2}\right). \tag{4}$$

In a similar fashion, the known relation for Chebychev polyonmials of the second kind,

$$U_n(x) = \tfrac{1}{2}\left[\frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{\sqrt{(x^2-1)}}\right], \tag{5}$$

gives, with $x = \sqrt{5}/2$, and n replaced by $2n - 1$,

$$F_{2n} = \frac{1}{\sqrt{5}} U_{2n-1}\left(\frac{\sqrt{5}}{2}\right), \quad n \geqslant 1. \tag{6}$$

The relation [8],

$$T_n(x) = U_n(x) - xU_{n-1}(x), \tag{7}$$

gives, after changing n to $2n + 1$, letting $x = \sqrt{5}/2$, and using (4) and (6) and the recurrence relation for F_n,

$$F_{2n} + F_{2n+2} = U_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{8}$$

Equation (7) gives, after changing n by $2n$, letting $x = \sqrt{5}/2$, using (4), (6), and the recurrence relation for F_n

$$\frac{F_{2n+1} + F_{2n-1}}{2} = T_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{9}$$

The relation [12]

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}, \tag{10}$$

which can be proved by induction, gives, after replacing both n and m by $2n+1$, together with (8), the result

$$\frac{F_{4n+2}}{F_{2n+1}} = U_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{11}$$

Replacing both n and m in (10) by $2n$ gives, together with (9), the result

$$\frac{F_{4n}}{F_{2n}} = 2T_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{12}$$

Equations (4), (6), (8) or (11), and (9) or (12) relate all Chebychev polynomials with argument $\sqrt{5}/2$ with Fibonacci numbers.

Identities that relate Chebychev polynomials lead to identities for Fibonacci numbers. For instance, the relation [5]

$$\sum_{m=0}^{n-1} T_{2m+1}(x) = \tfrac{1}{2}U_{2n-1}(x)$$

gives, with $x = \sqrt{5}/2$, the known result [6]

$$\sum_{m=0}^{n-1} F_{2m+1} = F_{2n}.$$

Equation [5]

$$2(1 - x^2)\sum_{m=1}^{n} U_{2m-1}(x) = x - T_{2n+1}(x),$$

gives, with $x = \sqrt{5}/2$, the known result [6]

$$\sum_{m=1}^{n} F_{2m} = F_{2n+1} - 1.$$

Binet's formula (3) gives us

$$\begin{aligned} F_{2n+1} &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n+1}\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n+1}\right] - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2n+1} \\ &= \frac{1}{\sqrt{5}}\, U_{2n}\left(\frac{\sqrt{5}}{2}\right) - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2n+1} \cong \frac{1}{\sqrt{5}}\, U_{2n}\left(\frac{\sqrt{5}}{2}\right). \end{aligned} \tag{13}$$

In an identical fashion, we obtain from (2) and (3) the approximation

$$F_{2n} \cong \frac{2}{\sqrt{5}}\, T_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{14}$$

Equations (8) and (9) combine with equations (13) and (14) to give the following interesting approximate relations

$$\frac{F_{n-1} + F_{n+1}}{\sqrt{5}} \cong F_n. \tag{15}$$

In (10), $m = n$ gives, together with (15), the following approximate relation:

$$\frac{F_{2n}}{\sqrt{5}} \cong F_n^2. \tag{16}$$

The exact relation corresponding to (15), obtainable from (3), is

$$\frac{F_{n-1} + F_{n+1}}{\sqrt{5}} = F_n + \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{17}$$

From (17) we see that (15) approximates F_n by excess for even n, and by defect for odd n.

Equations (13) to (16) give excellent approximations if n is greater than 5.

2. EXPANSIONS WITH FIBONACCI COEFFICIENTS

Chebychev polynomials are special cases of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. The exact relations are [5]:

$$T_n(x) = (g_n)^{-1} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x),$$

$$U_n(x) = (2g_{n+1})^{-1} P_n^{(\frac{1}{2},\frac{1}{2})}(x),$$

with

$$g_n = \frac{(\frac{1}{2})_n}{n!}.$$

Consider the expansion [8], due to Gegenbauer,

$$\exp(xt) = \left(\frac{t}{2}\right)^{-\nu} \Gamma(\nu) \sum_{n=0}^{\infty} (\nu + n) I_{\nu+n}(t) C_n^{\nu}(x), \tag{18}$$

where $I_k(t)$ are modified Bessel functions of the first kind [5], given by

$$I_k(t) = \frac{(\frac{1}{2}t)^k}{\Gamma(k+1)}\, {}_0F_1(-;\, 1+k;\, \tfrac{1}{4}t^2),$$

and $C_n^{\nu}(x)$ are ultraspherical polynomials [8] defined by:

$$C_n^{\nu}(x) = \frac{(2\nu)_n P_n^{(\nu-\frac{1}{2},\,\nu-\frac{1}{2})}(x)}{(\nu+\frac{1}{2})_n}$$

In terms of Gegenbauer polynomials, Chebychev polynomials are given by:

$$U_n(x) = C_n^1(x), \tag{19}$$

$$T_n(x) = \lim_{\nu\to 0} \frac{C_n^{\nu}(x)}{C_n^{\nu}(1)}. \tag{20}$$

In (18), replace x by $-x$, recalling that $C_n^{\nu}(-x) = (-1)^n C_n^{\nu}(x)$, and subtract the resulting series from (18) to obtain

$$\sinh xt = \left(\frac{t}{2}\right)^{-\nu} \Gamma(\nu) \sum_{n=1}^{\infty} (\nu + 2n + 1) I_{\nu+2n+1}(t) C_{2n+1}^{\nu}(x).$$

Now let $\nu = 1$. Replace t by $-it$ and recall that $I_n(-it) = i^{-n} J_n(t)$, where $J_n(t)$ are Bessel functions of the first kind. Let $x = \sqrt{5}/2$, replace n by $n-1$, and finally let $\sqrt{5}t/2 = \xi$, to obtain, with the help of (19) and (6),

$$\sin \xi = \frac{5}{\xi} \sum_{n=1}^{\infty} (-1)^{n+1} 2n F_{2n} J_{2n}(2\xi/\sqrt{5}). \tag{21}$$

Separating the *even* part of (18), instead of the odd, gives, with the use of (8) and (11),

$$\cos \xi = \frac{\sqrt{5}}{\xi} \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{F_{4n+2}}{F_{2n+1}} J_{2n+1}(2\xi/\sqrt{5}), \tag{22}$$

$$\cos \xi = \frac{5}{\xi} \sum_{n=0}^{\infty} (-1)^n (2n + 1)\left[\frac{F_{2n} + F_{2n+2}}{\sqrt{5}}\right] J_{2n+1}(2\xi/\sqrt{5}). \tag{23}$$

If use is made of equation (17), (23) can be written as

$$\cos \xi = \frac{5}{\xi} \sum_{n=0}^{\infty} (-1)^n (2n + 1) F_{2n+1} J_{2n+1}(2\xi/\sqrt{5})$$
$$+ \frac{2\sqrt{5}}{\xi} \sum_{n=0}^{\infty} (-1)^n (2n + 1)\left(\frac{1 - \sqrt{5}}{2}\right)^{2n+1} J_{2n+1}(2\xi/\sqrt{5}). \tag{24}$$

The terms in the second series in (24) tend to zero very rapidly with increasing n.

Series (21) to (24) converge very rapidly and are, to the author's knowledge, completely new results.

Paul Byrd [4] obtained some expressions for the sine and the cosine with Fibonacci coefficients that are very similar to (21) and (24). Byrd's results are:

$$\sin \xi = \frac{1}{\xi} \sum_{n=1}^{\infty} (-1)^{n+1} 2n F_{2n} I_{2n}(2\xi),$$

and

$$\cos \xi = \frac{1}{\xi} \sum_{n=0}^{\infty} (-1)^n (2n + 1) F_{2n+1} I_{2n+1}(2\xi),$$

where $I_n(\xi)$ are modified Bessel functions of the first kind.

From the series

$$\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

we obtain, in an obvious manner, using (3), the interesting expansion

$$\log\left[\frac{1 + \frac{1 + \sqrt{5}}{2}\, t}{1 + \frac{1 - \sqrt{5}}{2}\, t}\right] = \sqrt{5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} F_n t^n}{n}. \tag{25}$$

It must be noticed that this is a general technique. Given a function $f(x, t)$ that allows for an expansion of the form

$$f(x, t) = \sum_{n=0}^{\infty} a_n(t) C_n^{\nu}(x), \tag{26}$$

it is necessary only to give appropriate values to ν, and to let x equal $\sqrt{5}/2$, provided $\sqrt{5}/2$ is within the x-region of convergence, to come to an expression such as (21), (23), or (25). References [3] and [8] contain ample information on conditions that guarantee the validity of results such as (26).

It is important to bear in mind that Fibonacci numbers grow very rapidly, for example, $F_{10} = 55$, $F_{20} = 6765$, $F_{30} = 832{,}040$, $F_{40} = 102{,}334{,}155$. *Hence, when an expansion with Fibonacci coefficients is convergent, the* $a_n(t)$ *must decrease very rapidly with increasing* n. If t is not near the boundary of the t-region of convergence, this circumstance makes these series very amenable for numerical work. We will illustrate this fact in the following sections.

3. A SERIES FOR THE ARC TANGENT

Consider the identity,

$$\tan^{-1}\frac{2x\xi}{1-\xi^2} = \tan^{-1}[(x+\sqrt{x^2-1})\xi] + \tan^{-1}[(x-\sqrt{x^2-1})\xi],$$

easily verified by taking the tangent of both sides.

Let us substitute the expansion,

$$\tan^{-1}\xi = \sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{2n+1}, \tag{27}$$

on the two terms on the right-hand side above, and make use of equation (1) to obtain,

$$\tan^{-1}\frac{2x\xi}{1-\xi^2} = 2\sum_{n=0}^{\infty}\frac{(-1)^n T_{2n+1}(x)\xi^{2n+1}}{2n+1}. \tag{28}$$

Series (27), known as Gregory's series, is a special case of series (28) corresponding to $x = 1$, when use is made of the identity for the tangent of the half-angle: $\tan^{-1}[2\xi/(1-\xi^2)] = 2\tan^{-1}\xi$.

In (28), let $x = \sqrt{5}/2$, $\sqrt{5}\xi = t$, and use equation (4) to obtain

$$\tan^{-1}\frac{5t}{5-t^2} = \sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}t^{2n+1}}{5^n(2n+1)}.$$

Now let $5t/(5-t^2) = \alpha > 0$, and choose the smaller of the roots of this quadratic equation, to obtain

$$\tan^{-1}\alpha = \sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}t^{2n+1}}{5^n(2n+1)}, \tag{29}$$

with

$$t = \frac{2\alpha}{1+\sqrt{1+(4\alpha^2/5)}}, \tag{30}$$

a curious and simple series for the arc tangent with odd Fibonacci numbers as coefficients.

4. COMPARISON WITH EULER'S SERIES FOR THE ARC TANGENT

Series (27) discovered by Gregory in 1671, converges very slowly except for very small values of its argument. For $\xi = 1$, for example, it yields Leibniz' celebrates series for $\pi/4$ that requires two thousand terms to give three decimal figures of π.

Let us use Pochhammer's symbol

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), \quad \alpha \neq 0,$$

and the identity $(3/2)_n/(1/2)_n = 2n+1$ to write (27) in hypergeometric form:

$$\tan^{-1}t = tF(1, 1/2; 3/2; -t^2). \tag{31}$$

Now consider the relation

$$F(a,\ b;\ c;\ z) = (1 - z)^{-a}F(a,\ c - b;\ c;\ -z/(1 - z)), \tag{32}$$

valid if $|z| < 1$, and $|z/(1 - z)| < 1$. This relation is an equality among two of Kummer's twenty-four solutions to Gauss's hypergeometric differential equation. In (32), let $a = 1$, $b = 1/2$, $c = 3/2$, and $z = -t^2$, to obtain

$$\tan^{-1}t = tF(1,\ 1/2;\ 3/2;\ -t^2) = [t/(1 + t^2)]F(1,\ 1;\ 3/2;\ t^2/(1 + t^2)).$$

Since $(2n + 1)! = (2)_{2n} = 2^{2n}n!(3/2)_n$, the above equation gives

$$\tan^{-1}t = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n + 1)!} \frac{t^{2n+1}}{(1 + t^2)^{n+1}}. \tag{33}$$

Inasmuch as $t^2/(1 + t^2) < 1$ for every real t, we can conclude that (33) converges for every real value of its argument.

Equation (33) is Euler's famous series for the arc tangent discovered in 1755. This series converges very rapidly for all t, and especially for small values of its argument.

Let $t = \alpha \ll 1$. Using Stirling's formula for the factorial

$$n! \cong \sqrt{2\pi n}(n/e)^n, \quad n \text{ large},$$

we obtain, for the general term a_n of Euler's series, the estimate

$$a_n \cong \frac{e\sqrt{\pi}n^{2n+1}\alpha^{2n+1}}{2(n + \tfrac{1}{2})^{2n+3/2}}. \tag{34}$$

If n is large, $n + \tfrac{1}{2} \cong n$, and we have the estimate

$$a_n \cong \frac{e\sqrt{\pi}}{2} \frac{\alpha^{2n+1}}{\sqrt{n}}. \tag{35}$$

To compare this result with the corresponding one for series (29), notice that for α small (30) gives $t \cong \alpha$. For the general term, omitting the sign, b_n of series (29), we then have, recalling equation (3), the estimate

$$b_n \cong \frac{\alpha^{2n+1}}{2n + 1}\left(\frac{1 + 5^{-\frac{1}{2}}}{2}\right)^{2n+1}. \tag{36}$$

Comparison of (34) or (35) with (36), observing that the expression in parentheses above is $0.723606798\ldots < 1$, shows that, for small values of its argument, series (29) converges substantially faster than (33).

The requirement of the argument being small is only an apparent restriction, necessary to simplify the proof above. If α is large, it is simply necessary to use the identity

$$\tan^{-1}\alpha = \frac{\pi}{2} - \tan^{-1}\frac{1}{\alpha}, \quad (\alpha > 1).$$

Series (29) has the added advantage of being an alternating series, which series (33) is not. It is, as is well known, a general property of such series that the remainder after n terms has a value which is between zero and the first term not taken. It is a simple matter, then, to determine the number of terms of (29) needed to obtain a given accuracy.

If in series (28) we let $x = \cos\theta$, $2\xi\cos\theta/(1-\xi^2) = \alpha$, solve for ξ in terms of α and $\cos\theta$, and substitute back into (28), recalling that $T_n(\cos\theta) = \cos n\theta$, we obtain the curious series

$$\tan^{-1}\alpha = 2\sum_{n=0}^{\infty}\frac{(-1)^n\alpha^{2n+1}\cos(2n+1)\theta}{(2n+1)(\cos\theta+\sqrt{\alpha^2+\cos^2\theta})^{2n+1}}, \quad \left(0 \leqslant \theta < \frac{\pi}{2}\right), \tag{37}$$

where the right-hand side is independent of θ. The rapidity of the convergence, though, depends on the choice of θ. Series (37) converges very rapidly if both α and θ are small.

5. ANOTHER SERIES FOR THE ARC TANGENT

Iteration of the method used in Section 3 to obtain equation (28) yields a new series for the arc tangent. In (28), replace ξ by $\xi(x+\sqrt{x^2-1})$ and by $\xi(x-\sqrt{x^2-1})$ and add the two arc tangents to obtain

$$\tan^{-1}\frac{2x\xi(x+\sqrt{x^2-1})}{1-(x+\sqrt{x^2-1})^2\xi^2} + \tan^{-1}\frac{2x\xi(x-\sqrt{x^2-1})}{1-(x-\sqrt{x^2-1})^2\xi^2}$$

$$= 4\sum_{n=0}^{\infty}\frac{(-1)^nT^2_{2n+1}(x)\xi^{2n+1}}{2n+1}.$$

Combining the two arc tangents by means of the identity

$$\tan^{-1}a + \tan^{-1}b = \tan^{-1}\frac{a+b}{1-ab},$$

we obtain

$$\tan^{-1}\frac{4x^2\xi(1-\xi^2)}{1-2(4x^2-1)\xi^2+\xi^4} = 4\sum_{n=0}^{\infty}\frac{(-1)^nT^2_{2n+1}(x)\xi^{2n+1}}{2n+1}. \tag{38}$$

Gregory's series (27) is the special case of (38) corresponding to $x = 1$, if use is made of the identity for the tangent of one-fourth of the angle:

$$\tan^{-1}\xi = \frac{1}{4}\tan^{-1}\frac{4\xi(1-\xi^2)}{1-6\xi^2+\xi^4}.$$

Let the argument of the arc tangent in (38) equal α, and solve the resulting fourth-degree equation for ξ. The solution is easily obtained by dividing through by ξ^2, and making the substitutions $\xi - \xi^{-1} = -2t$, $\xi^2 + \xi^{-2} = 4t^2 + 2$, which reduces it to two quadratics. The results are

$$\xi = (t+\sqrt{t^2+1})^{-1} \quad\text{and}\quad t = \frac{x^2}{\alpha}(1+\sqrt{1+[\alpha^2(2x^2-1)/x^4]}). \tag{39}$$

Now, if we let $x = \sqrt{5}/2$ in (38) and (39), we obtain

$$\tan^{-1}\alpha = 5\sum_{n=0}^{\infty}\frac{(-1)^nF^2_{2n+1}}{(2n+1)(t+\sqrt{t^2+1})^{2n+1}}, \tag{40}$$

with

$$t = \frac{5}{4\alpha}(1 + \sqrt{1 + (24\alpha^2/25)}). \tag{41}$$

Series (40) converges substantially faster than series (29). If α is small, the n^{th} term, c_n, of series (40), is, apart from the sign, approximately given by

$$c_n \cong \frac{\alpha^{2n+1}}{2n+1}\left(\frac{1 + 5^{-\frac{1}{2}}}{2}\right)^{4n+2}. \tag{42}$$

6. THE NEXT ITERATION

Iteration of formula (38) by the method used in Section 5 gives, after some simple but lengthy algebra, the result

$$\tan^{-1}\frac{8x^3[\xi - (12x^2 - 5)\xi^3 + (12x^2 - 5)\xi^3 - \xi^7]}{1 - 4(12x^4 - 6x^2 + 1)\xi^2 + 2(32x^6 + 24x^4 - 24x + 3)\xi^4 - 4(12x^4 - 6x^2 + 1)\xi^6 + \xi^8}$$

$$= 8\sum_{n=0}^{\infty}\frac{(-1)^n T^3_{2n+1}(x)\xi^{2n+1}}{2n + 1}. \tag{43}$$

If we let the argument of the arc tangent in (43) equal α we obtain, after dividing through by ξ^4 and setting

$$\xi - \xi^{-1} = -2t/\sqrt{5},\ \xi^2 + \xi^{-2} = \frac{4}{5}t^2 + 2,\ \xi^3 - \xi^{-3} = -\frac{8}{5\sqrt{5}}t^3 - \frac{6}{\sqrt{5}}t,$$

$$\xi^4 + \xi^{-4} = \frac{16}{25}t^4 + \frac{16}{5}t^2 + 2,\ \text{and } x = \frac{\sqrt{5}}{2},$$

the result is

$$\tan^{-1}\alpha = \sum_{n=0}^{\infty}\frac{(-1)^n 5^{n+2}F^3_{2n+1}}{(2n + 1)(t + \sqrt{t^2 + 5})^{2n+1}}, \tag{44}$$

with t the largest positive root of

$$8\alpha t^4 - 100t^3 - 450\alpha t^2 + 875t + 625\alpha = 0. \tag{45}$$

This quartic equation is *in principle* solvable by radicals [1] for any value of α. The algorithm, though, does not seem to lead to any manageable combination of radicals, and for its solution we resorted to Newton's iterative method. Several solutions are discussed in the next section.

7. SOME SERIES FOR π

To illustrate the convergence of series (29), (40), and (44), we will obtain some expressions for π. Let $\alpha = 1$ in (30), and substitute into (29) to get

$$\pi = \sqrt{5}\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}2^{2n+3}}{(2n + 1)(3 + \sqrt{5})^{2n+1}}. \tag{46}$$

$\alpha = 1$ in equation (41) when substituted into (40) gives

$$\pi = 20 \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1}^2}{(2n+1)(3+\sqrt{10})^{2n+1}}. \tag{47}$$

Thirty-two terms of (46) give fifteen decimal places of π, while series (47) requires nineteen terms. For $\alpha = 1$, the largest positive root of (45) is

$$t = 15.63057705819013\ldots .$$

With this value of t fourteen terms of (44) give fifteen decimal figures of π. Euler's series (33) for the same argument and for the same accuracy requires fifty terms.

From equations (36) and (42), we see that rapid convergence of series (29) and (40) depends on our ability to choose the argument of the arc tangent, or, which is the same, the angle, sufficiently small. For example,

$$\frac{\pi}{12} = \tan^{-1}(2 - \sqrt{3}).$$

For this argument, series (29) gives

$$\pi = 12\sqrt{5} \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1}}{2n+1} \left[\frac{2(2-\sqrt{3})}{\sqrt{5} + \sqrt{1 + 16(2-\sqrt{3})}}\right]^{2n+1}. \tag{48}$$

The expression in brackets above is 0.1181577543..., which is more than seven times smaller than the corresponding root for equation (46). Series (48) converges very rapidly. Ten terms give fifteen decimal places of π. For this same argument the corresponding value of equation (41) is $t = 9.488217845\ldots$. With this value of t nine terms of (40) give fifteen decimal figures of π. Euler's series (33) for the same argument and for the same accuracy requires thirteen terms.

Use of Machin's formula,

$$\pi = 16 \tan^{-1}(1/5) - 4 \tan^{-1}(1/239), \tag{49}$$

with α in (41) equal to 1/5 and to 1/239 gives values for t which are approximately,

$$t = 12.61886960\ldots, \quad \text{and} \quad t' = 597.5025107\ldots .$$

Using these values on (49) with series (40), we obtain a very rapidly converging series for π. A computer run with the double-precision routines of the BASIC Level II interpreter of the Radio Shack TRS-80 Model I microcomputer with this combination of arc tangents gave the values shown in the following table for Gregory's series (27), Euler's series (33), and series (40). We see that series (40) consistently gives better approximations than either Euler's or Gregory's series. Series (29) will also, of course, converve more rapidly than Gregory's or Euler's series ($n = 9$).

n	Gregory	Euler	Series (40)
0	3.183263598326360	3.060186968243409	3.148158616418292
1	3.140597029326061	3.139082236428362	3.141554182069219
2	3.141621029325035	3.141509789149037	3.141592944101887
3	3.141591772182177	3.141589818359699	3.141592651171905
4	3.141591682404400	3.141592554401089	3.141592653611002
5	3.141592652615309	3.141592650066872	3.141592653589601
6	3.141592653623555	3.141592653463209	3.141592653589795
7	3.141592653588603	3.141592653585213	3.141592653589793
8	3.141592653589836	3.141592653589626	
9	3.141592653589792	3.141592653589787	
10	3.141592653589794	3.141592653589793	
11	3.141592653589793		

The largest positive root of (45) corresponding to $\alpha = 1/5$ is

$$t = 63.25229744727801\ldots,$$

and the one corresponding to $\alpha = 1/239$ is

$$t' = 2987.51589950963\ldots\ .$$

Using these values on (49) with series (44) gives fifteen decimal figures of π after seven terms ($n = 6$). We see that with the use of expressions such as Machin's identity (49), iterations beyond the second are not worth the added labor, it being much simpler to work with series (40).

Application of the trigonometric identities

$$\tan^{-1}\frac{1}{a \pm b} = \tan^{-1}\frac{1}{a} \mp \tan^{-1}\frac{b}{a^2 \pm ab + 1}, \tag{50}$$

and

$$\tan^{-1}\frac{1}{a} = 2\tan^{-1}\frac{1}{2a} - \tan^{-1}\frac{1}{4a^3 + 3a} \tag{51}$$

on simpler formulas such as Machin's identity, or on the identity

$$\pi = 20\tan^{-1}(1/7) + 8\tan^{-1}(3/79), \tag{52}$$

due to Euler, give additional expressions for the calculation of π. Repeated application of (51) to Machin's formula, letting a equal 5, 10, 20, and (40), in turn, yields the identity

$$\pi = 256\tan^{-1}(1/80) - 4\tan^{-1}(1/239) - 16\tan^{-1}(1/515) - 32\tan^{-1}(1/4030) - 64\tan^{-1}(1/32{,}060) - 128\tan^{-1}(1/256{,}120), \tag{53}$$

first obtained by Cashmore in [7]. Identity (53) together with (40) provides an extremely rapidly converging series for the calculation of π. Four terms of this series give fifteen decimal figures of π. Euler's series (33) also requires four terms. The computed values are shown in the following table.

n	Euler's Series	Series (40)
0	3.141259656493609	3.141619294232185
1	3.141592611940046	3.141592652964994
2	3.141592653584216	3.141592653589812
3	3.141592653589793	3.141592653589793

Once again we see that (40) converges to its limiting value faster than does Euler's series. As more decimal figures are calculated, though, the difference between the series becomes significant and the tide swings in favor of our series. For the same value of the argument, the tenth term of Euler's series is 4.54×10^6 times bigger than the tenth term of (40). The twentieth term of Euler's series is 2.60×10^{12} times bigger than the corresponding one of (40). The thirtieth term is 1.35×10^{18} times bigger. The one hundredth term is already 1.46×10^{58} times bigger, and the one hundred fiftieth term is 2.25×10^{86} times bigger.

With the combination of arc tangents given in (53), twenty-three terms of (40) give one hundred decimal places of π. Two hundred twenty-six terms will give one thousand decimal places of π. The calculation of the radicals in (40) and (41) can be performed very quickly, because of the smallness of α, with the quadratically converging algorithm given in Rudin [10]. Identity (53) is very amenable for a high-precision calculation of π. It would be of interest to compare (53) against Eugene Salamin's quadratically converging algorithm [11] based on the theory of elliptic integrals.

It should, perhaps, be mentioned that there exist series for the calculation of π which converge faster than any series we have obtained. For example,

$$\frac{1}{\pi} = 2\sqrt{2}\left[\frac{1103}{99^2} + \frac{27493}{99^6}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{53883}{99^{10}}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots\right], \tag{54}$$

due to Ramanujan [9]. The numerators of the first fractions of each term above are in arithmetic progression. Three terms of (54) give seventeen decimal figures of π!

As stated at the beginning of this section, we have used π simply as an illustration of the convergence of the arc tangent series (29), (40), and (44), and these series *do converge faster than any other known arc tangent series*.

It is an interesting historical fact that Fibonacci made an attempt to determine the value of π using Archimedes' method of inscribed and circumscribed polygons. Using a 96-sided polygon, he obtained for π the approximation $864 \div 275$, which gave him the value 3.141818, correct to three decimal places [2]. It seems safe to think that he never suspected that the peculiar sequence he had discovered on the growth of the rabbit population would yield, nearly eight centuries later, a simple and powerful algorithm for the calculation of π with any desired accuracy.

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