GENERALIZED ZIGZAG POLYNOMIALS

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1. INTRODUCTION

The purpose of this paper is to extend and generalize the results established in [5] for a category of polynomials described therein as "zigzag." These arise in a specified way from a given polynomial sequence generated by a second-order recurrence relation.

Consider the sequence of polynomials $\{W_n(x)\}$ defined by the second-order recurrence relation

$$W_{n+2}(x) = kxW_{n+1}(x) + mW_n(x) \qquad (n \geqslant 0) \tag{1.1}$$

with initial values

$$W_0(x) = h, \quad W_1(x) = kx, \tag{1.2}$$

wherein h, k, and m are real numbers, usually integers.

We have represented these polynomials in abbreviated form by $W_n(x)$ though the parametric symbolism $W_n(h, kx; kx, m)$ more fully describes them. Note that a characteristic feature of the definition (1.1) and (1.2) is that the initial value $W_1(x) = kx$ in (1.2) must be the same as the coefficient of $W_{n+1}(x)$ in the recurrence (1.1).

Standard methods enable us to derive the generating function for $\{W_n(x)\}$, namely,

$$\sum_{n=0}^{\infty} W_n(x)t^n = \{h + kx(1 - h)t\}[1 - (kxt + mt^2)]^{-1} \tag{1.3}$$

or, equivalently,

$$\sum_{n=0}^{\infty} W_{n+1}(x)t^n = (kx + mht)[1 - (kxt + mt^2)]^{-1}. \tag{1.3$'$}$$

An explicit form of $W_n(x)$ $(n \geqslant 2)$ is, in the usual notation,

$$W_n(x) = kx \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-i}{i} m^i (kx)^{n-1-2i} + mh \sum_{i=0}^{\left[\frac{n-2}{2}\right]} \binom{n-2-i}{i} m^i (kx)^{n-2-2i}. \tag{1.4}$$

This formula will be essential when we prove (3.3).

At this point, we stress that $W_n(h, kx; kx, m)$ defined above is a polynomial variation of the $W_n(a, b; p, q)$, wherein $a = h$, $b = p = kx$, $q = m$, whose basic and special properties have been discussed in [7] and [8]. Therefore, no further consideration of its salient features is required here.

Special cases of $W_n(h, kx; kx, m)$ which interest us are (when $h = 2$):

$$\left\{\begin{array}{llll}
\text{POLYNOMIALS} & h & k & m \\
\textit{Lucas} & 2 & 1 & 1 \\
\textit{Pell-Lucas} & 2 & 2 & 1 \\
\textit{Chebyshev (2nd kind)} & 2 & 2 & -1 \\
\textit{Fermat} & 2 & 1 & -2
\end{array}\right. \tag{1.5}$$

More will be said about these special cases in Section 4.

2. RISING DIAGONAL ZIGZAG POLYNOMIALS

The first few members of the polynomial set $\{W_n(x)\}$ are, from (1.1) with (1.2):

Table 1. Rising Diagonal Zigzag Polynomials for $\{W_n(x)\}$

$$Z_0(x),\ Z_1(x),\ Z_2(x),\ Z_3(x),\ Z_4(x),\ Z_5(x),\ Z_6(x),\ Z_7(x),\ Z_8(x),\ Z_9(x),\ Z_{10}(x)$$

$$\left\{\begin{aligned}
W_0(x) &= h \\
W_1(x) &= kx \\
W_2(x) &= (kx)^2 + mh \\
W_3(x) &= (kx)^3 + mh(kx) + m(kx) \\
W_4(x) &= (kx)^4 + mh(kx)^2 + 2m(kx)^2 + m^2h \\
W_5(x) &= (kx)^5 + mh(kx)^3 + 3m(kx)^3 + 2m^2h(kx) + m^2(kx) \\
W_6(x) &= (kx)^6 + mh(kx)^4 + 4m(kx)^4 + 3m^2h(kx)^2 + 3m^2(kx)^2 + m^3h \\
W_7(x) &= (kx)^7 + mh(kx)^5 + 5m(kx)^5 + 4m^2h(kx)^3 + 6m^2(kx)^3 + 3m^3h(kx) + m^3(kx) \\
W_8(x) &= (kx)^8 + mh(kx)^6 + 6m(kx)^6 + 5m^2h(kx)^4 + 10m^2(kx)^4 + 6m^3h(kx)^2 + 4m^3(kx)^2 + m^4h \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right. \tag{2.1}$$

In Table 1, pair terms in columns 2 and 3, columns 4 and 5, ..., to form the *rising diagonal generalized zigzag polynomials* $Z_n(x)$ as indicated by the lines, beginning with $Z_0(x) = h$. For example, some of these generalized zigzag polynomials are:

$$\left\{\begin{array}{l}
Z_0(x) = h,\ Z_1(x) = kx,\ Z_2(x) = (kx)^2,\ Z_3(x) = (kx)^3 + mh, \\
Z_4(x) = (kx)^4 + mk(kx) + m(kx),\ Z_5(x) = (kx)^5 + mh(kx)^2 + 2m(kx)^2, \\
Z_6(x) = (kx)^6 + mh(kx)^3 + 3m(kx)^3 + m^2h,\ \ldots .
\end{array}\right. \tag{2.2}$$

Previously, in [5], we mentioned that the virtue of the pairing technique by which the zigzag polynomials are produced is that specializations may be readily obtained. In the case of Table 1 this is achieved by the amalgamation of corresponding elements in appropriate pairs of columns.

For example, the rising diagonal polynomials for Pell-Lucas polynomials (1.5), already given in [5], are obtained by adding like terms in columns 2 and 3, columns 4 and 5, ... (as appropriate), in Table 1 when $h = 2$, $k = 2$, $m = 1$, to give, for instance, the special expression for $Z_6(x)$ in (2.2) as

$$64x^6 + 40x^3 + 2$$

(which is the polynomial $r_6(x)$ in [5]).

Correspondingly, for the Fermat polynomials (1.5) the rising diagonal polynomial is $x^6 - 10x^3 + 8$ (represented in [3] by $R'(x)$).

Before proceeding to establish some properties of $Z_n(x)$, we introduce the companion polynomials $X_n(x)$, defined by

$$X_n(x) = Z_n(x)\Big|_{h=1}, \tag{2.3}$$

i.e., $X_n(x)$ are the rising diagonal zigzag polynomials of the set of polynomials $\{W_n(x)\}$ defined in (1.1) for which $h = 1$.

Thus, if we consider the four special cases of $W_n(2, kx; kx, m)$ which are listed in (1.5), yielding particular instances of the $Z_n(x)$ when $h = 2$ [the polynomials $Y_n(x)$ defined in (2.11) below), then the corresponding polynomials $X_n(x)$ are associated with the four special cases of $W_n(1, kx; kx, m)$ corresponding to those in (1.5), but with $h = 1$. These are the *Fibonacci polynomials*, the *Pell polynomials*, the *Chebyshev polynomials of the first kind*, and the *companion Fermat polynomials* ("Fermat polynomials of the first kind"), respectively.

From (2.2) and (2.3) we have the expressions for the simplest polynomials $X_n(x)$:

$$\begin{cases} X_0(x) = 1,\ X_1(x) = kx,\ X_2(x) = (kx)^2,\ X_3(x) = (kx)^3 + m, \\ X_4(x) = (kx)^4 + 2m(kx),\ X_5(x) = (kx)^5 + 3m(kx)^2, \\ X_6(x) = (kx)^6 + 4m(kx)^3 + m^2,\ \ldots\ . \end{cases} \tag{2.4}$$

The recurrence relation, the generating function, and the explicit form for $X_n(x)$ corresponding to (2.5)-(2.7), and the differential equations corresponding to (2.8) and (2.9) which $X_n(x)$ satisfy, may all be readily derived by simple substitution.

Following procedures already established in [5], we derive, without much effort, the results exhibited below.

RECURRENCE RELATION

$$Z_n(x) = kxZ_{n-1}(x) + mZ_{n-3}(x) \qquad (n \geqslant 3) \tag{2.5}$$

GENERATING FUNCTION

$$\sum_{n=1}^{\infty} Z_n(x)t^{n-1} = (kx + mht^2)[1 - (kxt + mt^3)]^{-1} \equiv Z(x, t) \tag{2.6}$$

EXPLICIT FORM

$$Z_n(x) = kx\sum_{i=0}^{\left[\frac{n-1}{3}\right]}\binom{n-1-2i}{i}m^i(kx)^{n-1-3i} + mh\sum_{i=0}^{\left[\frac{n-3}{3}\right]}\binom{n-3-2i}{i}m^i(kx)^{n-3-3i} \qquad (n \geqslant 3) \tag{2.7}$$

DIFFERENTIAL EQUATIONS

$$kt\frac{\partial}{\partial t}Z(x, t) - (kx + 3mt^2)\frac{\partial}{\partial x}Z(x, t) = k\{(2h-3)mt^2 - kx\}[1 - (kxt + mt^3)]^{-1} \tag{2.8}$$

$$kx\frac{d}{dx}Z_{n+2}(x) + 3m\frac{d}{dx}Z_n(x) = k\{(n-1)Z_{n+2}(x) + 3X_{n+2}(x)\} \tag{2.9}$$

Alternative and equivalent forms exist in some of the above results. For example, the bracketed factor on the right-hand side of (2.9) may be equally well expressed as

$$(n + 2)Z_{n+2}(x) - 3m(h - 1)X_{n-1}(x).$$

The equality of these two forms arises from the relationship

$$Z_n(x) = X_n(x) + m(h - 1)X_{n-3}(x) \qquad (n \geqslant 3), \tag{2.10}$$

which may be readily demonstrated. Substitution of $h = 1$ in (2.10) produces $Z_n(x) = X_n(x)$, of course, in accord with (2.3).

Another alternative expression occurs in the right-hand side of (2.8), which can be made to simplify to $k\{2Z(x, t) - 3X(x, t)\}$ where the symbol

$$X(x, t) = Z(x, t)\Big|_{h=1}.$$

Next, for completion, we introduce the related polynomial $Y_n(x)$, defined by

$$Y_n(x) = Z_n(x)\Big|_{h=2}, \tag{2.11}$$

i.e., the $Y_n(x)$ are the particular cases of $Z_n(x)$ occurring when $h = 2$.

Expressions for some of the $Y_n(x)$ are, by (2.2) and (2.11):

$$\begin{cases} Y_0(x) = 2,\ Y_1(x) = kx,\ Y_2(x) = (kx)^2,\ Y_3(x) = (kx)^3 + 2m, \\ Y_4(x) = (kx)^4 + 3m(kx),\ Y_5(x) = (kx)^5 + 4m(kx)^2, \\ Y_6(x) = (kx)^6 + 5m(kx)^3 + 2m^2,\ \ldots, \end{cases} \tag{2.12}$$

whence, by (2.4) and (2.12),

$$Y_n(x) = X_n(x) + mX_{n-3}(x). \tag{2.13}$$

Corresponding to (2.5)-(2.9), the recurrence relation, the generating function, and the explicit form for $Y_n(x)$, along with the differential equations satisfied by $Y_n(x)$, are easily deducible.

Subtraction of (2.13) from (2.10) reveals that

$$Z_n(x) = Y_n(x) + m(h - 2)X_{n-3}(x). \tag{2.14}$$

When $h = 2$, (2.14) leads to $Z_n(x) = Y_n(x)$ in accord with (2.11).

3. DESCENDING DIAGONAL ZIGZAG POLYNOMIALS

Re-organize the material in Table 1, as indicated in Table 2 below, to produce the *descending diagonal generalized zigzag polynomials*:

Table 2. Descending Diagonal Zigzag Polynomials for $\{W_n(x)\}$

$$\left\{\begin{array}{lll}
W_0(x) &=& h \qquad\qquad z_0(x) \\
W_1(x) &=& kx \\
W_2(x) &=& (kx)^2 + mh \;\text{———}\; z_1(x) \\
W_3(x) &=& (kx)^3 + mh(kx) \;+\; m(kx) \\
W_4(x) &=& (kx)^4 + mh(kx)^2 + 2m(kx)^2 + m^2h \;\text{———}\; z_2(x) \\
W_5(x) &=& (kx)^5 + mh(kx)^3 + 3m(kx)^3 + 2m^2h(kx) \;+\; m^2(kx) \\
W_6(x) &=& (kx)^6 + mh(kx)^4 + 4m(kx)^4 + 3m^2h(kx)^2 + 3m^2(kx)^2 + m^3h \;\text{———}\; z_3(x) \\
W_7(x) &=& (kx)^7 + mh(kx)^5 + 5m(kx)^5 + 4m^2h(kx)^3 + 6m^2(kx)^3 + 3m^3h(kx) \;+\; m^3(kx) \\
W_8(x) &=& (kx)^8 + mh(kx)^6 + 6m(kx)^6 + 5m^2h(kx)^4 + 10m^2(kx)^4 + 6m^3h(kx)^2 + 4m^3(kx)^2 + m^4h \\
\multicolumn{3}{l}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}\right. \tag{3.1}$$

Designate these polynomials by $z_n(x)$. Then, as we learned from experience to expect, we derive the relatively simple expressions

$$\begin{cases} z_0(x) = h,\ z_1(x) = kx + mh,\ z_2(x) = (kx + mh)(kx + m), \\ z_3(x) = (kx + mh)(kx + m)^2,\ z_4(x) = (kx + mh)(kx + m)^3,\ \ldots, \end{cases} \tag{3.2}$$

and in general

$$z_n(x) = (kx + mh)(kx + m)^{n-1} \qquad (n = 1, 2, 3, \ldots), \tag{3.3}$$

so that

$$\frac{z_{n+1}(x)}{z_n(x)} = kx + mh. \tag{3.4}$$

As result (3.3) is crucial, we proceed to demonstrate its validity.

Proof of (3.3): Temporarily, write $W_n(x) = kxP(x) + mhQ(x)$ in (1.4), wherein $P(x)$ and $Q(x)$ stand for the appropriate summations.

Let typical values of i in $P(x)$ and $Q(x)$ be represented by p and q respectively ($p = 0, 1, \ldots, n - 1$; $q = 0, 1, \ldots, n - 1$).

Each value of n in the $W_n(x)$ giving rise to a specified $z_n(x)$ in Table 2 requires a pair of values (p, q).

For

$$W_n(x),\ W_{n+1}(x),\ W_{n+2}(x),\ \ldots,\ W_{2n-1}(x),\ W_{2n}(x),$$

these are

$$(0, -),\ (1, 0),\ (2, 1),\ \ldots,\ (n - 1, n - 2),\ (-, n - 1),$$

respectively, in which the dash (-) signifies nonoccurrence.

Then, from (1.4), we have, after the necessary simplifications:

$$
\begin{aligned}
z_n(x) &= kx\left\{\binom{n-1}{0}(kx)^{n-1}m^0 + \binom{n-1}{1}(kx)^{n-2}m^1 + \binom{n-1}{2}(kx)^{n-3}m^2 + \cdots \right. \\
&\quad \left. + \binom{n-1}{n-1}(kx)^0 m^{n-1}\right\} + mh\left\{\binom{n-1}{0}(kx)^{n-1}m^0 + \binom{n-1}{1}(kx)^{n-2}m^1 \right. \\
&\quad \left. + \binom{n-1}{2}(kx)^{n-3}m^2 + \cdots + \binom{n-1}{n-1}(kx)^0 m^{n-1}\right\} \\
&= kx(kx+m)^{n-1} + mh(kx+m)^{n-1} \\
&= (kx+mh)(kx+m)^{n-1}.
\end{aligned}
$$

The generating function for $z_n(x)$ $(n > 0)$ is

$$z(x,\ t) \equiv \sum_{n=1}^{\infty} z_n(x)t^{n-1} = (kx+mh)[1-(kx+m)t]^{-1}. \tag{3.5}$$

Differential equations satisfied by the descending diagonal zigzag polynomials are, from (3.3) and (3.5),

$$kt\frac{\partial}{\partial t}z(x,\ t) - (kx+m)\frac{\partial}{\partial x}(x,\ t) + k\frac{(kx+m)}{(kx+mh)}z(x,\ t) = 0 \tag{3.6}$$

and

$$(kx+m)\frac{d}{dx}z_n(x) - k(n-1)z_n(x) - k(kx+m)^n = 0. \tag{3.7}$$

Just as we have the specialized forms (2.3) and (2.11) of $Z_n(x)$ occurring when $h = 1$ and $h = 2$ respectively, so we have the specialized forms of $z_n(x)$:

$$x_n(x) = z_n(x)\Big|_{h=1} \tag{3.8}$$

and

$$y_n(x) = z_n(x)\Big|_{h=2}. \tag{3.9}$$

Consequently,

$$x_n(x) = (kx+m)^n \tag{3.10}$$

and

$$y_n(x) = (kx+2m)(kx+m)^{n-1}. \tag{3.11}$$

Result (3.7) may then, by (3.10), have the factor of k in the last term replaced by $x_n(x)$.

Obviously, (3.10) and (3.11) together yield

$$\frac{x_n(x)}{y_n(x)} = \frac{kx+m}{kx+2m} \tag{3.12}$$

and

$$y_n(x) = mx_{n-1}(x) + x_n(x). \tag{3.13}$$

4. SPECIAL CASES

Recall that our generalization in this paper relates specifically to the situations in which

$$W_1(x) = kx = \text{the coefficient of } W_{n+1}(x) \text{ in the definition (1.1).}$$

This leads to some interesting and familiar polynomials which have been listed in (1.5).

Details concerning the results for the rising and descending diagonal polynomials cataloged in (1.5) are to be found in a chain of papers in the following sources:

POLYNOMIAL	REFERENCE
Lucas	[2]
Pell-Lucas	[5]
Chebyshev	[1], [3], [10]
Fermat	[3], [4]

(4.1)

where the reference numbers are those in the bibliographical references below.

Results for these specialized polynomials should be compared with the corresponding generalized results in this paper. Allowance must, however, be duly made on occasion for slight variations in notation, especially where these involve the initial conditions.

These principles are now carefully illustrated for the case of the Fermat polynomials ("of the second kind") in (1.5) for which $h = 2$. The companion Fermat polynomials ("of the first kind") for which $h = 1$ will also be required. In the illustration, we verify that equation (2.8) above does indeed reduce to equation (39) in [4] for the Fermat polynomials.

Illustration (Fermat Polynomials): For the Fermat polynomials we have, by substitution in (2.6),

$$\begin{aligned} Y = Y(x, t) &= (x - 4t^2)[1 - (xt - 2t^3)]^{-1} \\ &= Y_1(x) + Y_2(x)t + Y_3(x)t^2 + \cdots, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} X = X(x, t) &= (x - 2t^2)[1 - (xt - 2t^3)]^{-1} \\ &= X_1(x) + X_2(x)t + X_3(x)t^2 + \cdots, \end{aligned} \tag{4.3}$$

using a simplified notation.

Now in [3] and [4] the following notation was employed [wherein the dash (') does not indicate differentiation]:

$$R = [1 - (xt - 2t^3)]^{-1} = R_1(x) + R_2(x)t + R_3(x)t^2 + \cdots \equiv R(x, t); \tag{4.4}$$

$$R' = (1 - 2t^3)[1 - (xt - 2t^3)]^{-1} = 1 + R_2'(x)t + R_3'(x)t^2 + \cdots \equiv R'(x, t). \tag{4.5}$$

But

$$X_n(x) = R_{n+1}(x) \tag{4.6}$$

and

$$Y_n(x) = R'_{n+1}(x). \tag{4.7}$$

Hence (4.2)-(4.7) give

$$X = \frac{R - 1}{t} \tag{4.8}$$

and

$$Y = \frac{R' - 1}{t}.$$

Substitution in (2.8) from (1.5) for Fermat polynomials leads to

$$t\frac{\partial Y}{\partial t} - (x - 6t^2)\frac{\partial Y}{\partial x} = (-x - 2t^2)[1 - (xt - 2t^3)]^{-1},$$

i.e., by (4.9),

$$t\left(\frac{1}{t}\frac{\partial R'}{\partial t} - \frac{R' - 1}{t^2}\right) - (x - 6t^2)\frac{\partial R'}{\partial x} = (-x - 2t^2)[1 - (xt - 2t^3)]^{-1},$$

$$t\frac{\partial R'}{\partial t} - (x - 6t^2)\frac{\partial R'}{\partial x} = t(-x - 2t^2)[1 - (xt - 2t^3)]^{-1} + R' - 1$$

$$= -6t^2[1 - (xt - 2t^3)]^{-1}$$

$$= 3(R' - R) \qquad \text{by (4.4) and (4.5).}$$

This is equation (39) in [4], which we set out to verify.

In addition to the comments preceding the illustration, we remark that corresponding properties are developed for the polynomials $W_n(2, px; px, q)$ in [4], while in [6] and [9] analogous properties of the Gegenbauer polynomials, which are closely related to the Chebyshev polynomials, are investigated. (Brief mention is also made in [4] of the generalized Humbert polynomial of which the Gegenbauer and Chebyshev polynomials are particular cases.)

Some interesting number sequences result if appropriate values of x (e.g., $x = \frac{1}{2}$, $x = 1$) are substituted in the various rising and descending diagonal polynomials discussed in the above papers.

Thus, we have presented a summary and a synthesis of the basic thrust of the material in papers [1]-[6] and [9] by the author, along with that in [10] by Jaiswal.

5. POSSIBLE EXTENSIONS

One would like to be able to extend some of the ideas which have been applied in this paper to recurrence relations of higher order, particularly to the case of third-order recurrence relations. In order to produce the most worthwhile results, it would be necessary to choose the most fertile initial polynomials (including constants) to generate the required polynomial set.

Given such a fruitful selection of initial conditions, it might be possible to discover some geometrical results in three dimensions (Euclidean space) which would be analogous to, or extensions of, similar results about circles (in the Euclidean plane) by the author in other papers which are not listed in the References. These investigations could be extended to three-dimensional surfaces corresponding to the conics in the plane.

Hopefully (if tediously), such considerations could be further extended to hyper-surfaces in multi-dimensional Euclidean space.

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