THE FIRST DIGIT PROPERTY FOR EXPONENTIAL SEQUENCES IS INDEPENDENT OF THE UNDERLYING DISTRIBUTION

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The natural density in the set $R \equiv \{cr^k : k = 0, 1, 2, \ldots\}$, where $c > 0$, $r > 1$, and $\log_{10} r$ is irrational, of the elements beginning with the first digit ℓ is known to be

$$\log_{10}\left(\frac{1+\ell}{\ell}\right).$$

We show that this property persists for *any* finitely additive, translation invariant density on sets of the form

$$E \equiv \{e_k \equiv (cr^k + a_k) : a_k = o(r^k),\ k = 0, 1, \ldots\},$$

where $c > 0$ and $\log_{10} r$ is irrational.

In particular, this includes the Fibonacci sequences.

Let c and r be real numbers, such that $c > 0$ and $r > 1$, but $r \neq 10^q$ for q a rational number. Define

$$R \equiv \{cr^k : k = 0, 1, 2, \ldots\}$$

and let $R(\ell)$ be the subset of R whose members begin with the string of digits ℓ in the decimal representation, e.g., if $c = 3$ and $r = 7$, then $147 \in R(1)$ (147 begins with digit 1); 147 is also in $R(14)$ (147 begins with a two-digit string 14), and $147 \in R(147)$. If A is any subset of R, define its indicator function as follows:

$$\chi(k;\, A) = \begin{cases} 1 & \text{if } cr^{k-1} \in A \\ 0 & \text{if } cr^{k-1} \notin A \end{cases} \qquad k = 1, 2, 3, \ldots .$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi(k;\, R(\ell)) = \log_{10}\left(\frac{1+\ell}{\ell}\right),$$

which is a consequence of the fact that the set

$$\{(\log_{10} cr^k) \bmod 1 : k = 0, 1, 2, \ldots\}$$

is uniformly distributed in the interval $[0, 1)$. (See [4].)

When the limit exists,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi(k;\, A)$$

is called the natural density of A with respect to R. Although the natural density exists for each $R(\ell)$, there are subsets of R which do not have natural density. Nevertheless, the natural density can be extended to all subsets of R in a way which preserves finite additivity and translation invariance [defined below as properties (D1) and (D2)]. However, even with added restrictions such as scale invariance, such extensions are not unique. (See [1].)

Now consider any density d on R which satisfies the following two properties:

(D1) For all $A,\ B \subset R$, $d(A \cup B) = d(A) + d(B) - d(A \cap B)$ (finite additivity).

(D2) For all $A \subset R$, $d(A) = d(A^+)$, where A^+ is the "successor set" defined by $A^+ \equiv \{cr^k : cr^{k-1} \in A\}$ (translation invariance).

Subsequent successor sets to A will be denoted by

$$A^{+h} = (A^{+h-1})^+ = \{cr^k : cr^{k-h} \in A\}.$$

Notice that $A^+ = rA$ and $A^{+h} = r^hA$. Note also that (D2) implies that $d(A) = d(A^{+h})$ for all $h = 2, 3, 4, \ldots$, and that $d(A) = 0$ if A is finite [since $d(R) = 1$].

Naturally, the natural density satisfies (D1) and (D2).

We remark that any density defined on an algebra of subsets of R which includes the single point sets, $\{cr^k\}$ for each $k = 0, 1, 2, \ldots$, and which satisfies (D1) and (D2), can be extended to all subsets of R. We presume that any density considered in Theorems I and II is defined on the entire power set. Also, since finite sets and sets of density zero are unimportant in the sequel, we adopt the following definitions:

If $A,\ B \subset R$, say

(i) $A =_d B$ if and only if $d(A) = d(A \cap B) = d(B)$, and
(ii) $A \subset_d B$ if and only if $d(A) = d(A \cap B) \leqslant d(B)$.

Theorem I: For any density d on R which satisfies properties (D1) and (D2),

$$d(R(\ell)) = \log_{10}\left(\frac{1+\ell}{\ell}\right).$$

Proof of Theorem I: There are two key observations to be made about the first digit sets, $R(\ell)$. The first observation is that

$$\begin{aligned} R(1) &=_d R(10) \cup R(11) \cup R(12) \cup \cdots \cup R(19) \\ &=_d R(100) \cup R(101) \cup \cdots \cup R(199), \end{aligned}$$

$$\begin{aligned} R(2) &=_d R(20) \cup R(21) \cup \cdots \cup R(29) \\ &=_d R(200) \cup R(201) \cup \cdots \cup R(299), \end{aligned}$$

etc. Since $R = R(1) \cup R(2) \cup \cdots \cup R(9)$ and $R(j) \cap R(\ell) = \emptyset$ for $1 \leqslant j < \ell \leqslant 9$, it follows that

$$\sum_{j=10^k}^{10^{k+1}-1} d(R(j)) = 1 \quad \text{for } k = 0, 1, 2, \ldots . \tag{1}$$

The second key observation concerns the successor sets of the first digit sets. In the case in which c and r are integers, they have the form:

$$R(1)^+ =_d R(r) \cup R(r+1) \cup \cdots \cup R(2r-1)$$
$$R(2)^+ =_d R(2r) \cup R(2r+1) \cup \cdots \cup R(3r-1)$$
$$R(\ell)^+ =_d \bigcup_{j=\ell r}^{(\ell+1)r-1} R(j). \tag{2}$$

Then

$$d(R(\ell)) = \sum_{j=\ell r}^{(\ell+1)r-1} d(R(j)) \quad \text{for } \ell = 1, 2, 3, \ldots . \tag{3}$$

The idea of the proof is to tie together formula (1) and formula (3). However, if the decimal expansion of r does not terminate, $R(r)$ is no longer a well-defined object; thus, before proceeding further, it is necessary to generalize the notion of first digit sets.

If $1 \leqslant x \leqslant y \leqslant 10x$, define

$$R(x, y) \equiv \{u \in R : x \leqslant 10\ u < y \text{ for some integer } j\}.$$

Note that $R(\ell) = R(\ell, \ell+1)$.

For notational simplicity, assume $r < 10$. Otherwise, in what follows replace r by $\overline{r}$, defined by

$$\overline{r} \equiv r10^{-[\log_{10} r]},$$

where the brackets denote the greatest integer function, e.g., $[3.76] = 3$. Then

$$R(1, r)^+ =_d R(r, r^2), \quad R(1, r)^{+h} =_d R(r^h, r^{h+1}),$$

and equation (2) generalizes to

$$R(x, y)^+ = R(xr, yr). \tag{4}$$

By assumption (D2) of translation invariance,

$$md(R(1, r)) = \sum_{h=0}^{m-1} d(R(1, r)^{+h}) = \sum_{h=0}^{m-1} d(R(r^h, r^{h+1})). \tag{5}$$

By assumption (D1) of finite additivity, and the fact that $r < 10$,

$$d(R(1, r)) + d(R(r, r^2)) + \cdots + d(R(r^{m-1}, r^m))$$
$$= \sum_{\ell=1}^{[r^m]-1} d(R(\ell)) + d(R([r^m], r^m)). \tag{6}$$

Combining equations (1), (5), and (6) yields

$$[md(R(1, r))] = \sum_{\ell=1}^{[r^m]-1} d(R(\ell)) + d(R([r^m], r^m)) = [m \log_{10} r]. \tag{7}$$

Since equation (7) must be true for any choice of m, it follows that

$$d(R(1, r)) = \log_{10} r.$$

Now let $1 \leqslant x \leqslant 10$. We show that $d(R(1, r)) = d(R(x, xr))$.

Case 1: $1 < r \leqslant x \leqslant 10$.

$$d(R(1, x)) = d(R(1, r)) + d(R(r, x))$$

and

$$d(R(r, rx)) = d(R(r, x)) + d(R(x, xr)).$$

By (D2), $d(R(1, x)) = d(R(r, rx))$, so the result follows.

Case 2: $1 \leqslant x \leqslant r < 10$.

Again using $d(R(1, x)) = d(R(r, rx))$, we have

$$\begin{aligned} d(R(1, r) &= d(R(1, x)) + d(R(x, r)) \\ &= d(R(r, rx)) + d(R(x, r)) = d(R(x, rx)). \end{aligned}$$

Hence, by repeated use of (D2),

$$\log_{10} r = d(R(1, r)) = d(R(xr^j, xr^{j+1})) \quad \text{for any } j \geqslant 0,$$

so that

$$md(R(1, r)) = \sum_{j=0}^{m-1} d(R(xr^j, xr^{j+1})),$$

from which it follows that

$$md(R(1, r)) + d(R(1, x)) = \sum_{\ell=1}^{[xr^m]-1} d(R(\ell)) + d(R([xr^m], xr^m)),$$

which implies

$$[m \log_{10} r + d(R(1, x))] = [m \log_{10} r + \log_{10} x].$$

Thus

$$d(R(1, x)) = \log_{10} x. \tag{8}$$

Since $d(R(x, y)) = d(R(10^j x, 10^j y))$ by the definition of $R(x, y)$, for all integers j, the results

$$d(R(x, y)) = \log_{10}(y/x) \quad \text{for } 1 \leqslant x \leqslant y \leqslant 10x$$

and

$$d(R(\ell)) = \log_{10}\left(\frac{\ell + 1}{\ell}\right)$$

follow easily from equation (8) and assumption (D1). Q.E.D.

Now consider real numbers c and r as above and real numbers a_k for $k = 0$, 1, 2, ..., such that $a_k = o(r^k)$. Define

$$E \equiv \{e_k \equiv (cr^k + a_k) : k = 0, 1, 2, \ldots\},$$

and a corresponding set

$$R_E \equiv \{(e_k - a_k) : k = 0, 1, 2, \ldots\}.$$

Define a bijective function $f: E \to R_E$ by

$$f(e_k) \equiv e_k - a_k = cr^k.$$

Let the sets $E(x, y)$, $E(\ell)$, $R_E(x, y)$, $R_E(\ell)$ be defined as above.

Assumptions (D1) and (D2), and the notions of a successor set, $=_d$, and $\subset_d$ all extend to E in a natural fashion (although it is no longer true that $A^+ = rA$ for the successor set of $A \subset E$). Sets of type E include linear recursive sequences of the form

$$w_{n+1} = \alpha_0 w_n + \alpha_1 w_{n-1} + \cdots + \alpha_k w_{n-k}$$

whenever the characteristic equation has a unique highest root. In particular, the classic Fibonacci numbers $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$ occur when

$$c = \frac{1}{\sqrt{5}}, \quad r = \frac{1 + \sqrt{5}}{2}, \quad a_k = -\frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^k.$$

Note that $\log_{10}\left(\frac{1 + \sqrt{5}}{2}\right)$ is indeed irrational.

Theorem II: Let d be a density on E satisfying assumptions (D1) and (D2), as they extend to E. Then

$$d(E(\ell)) = \log_{10}\left(\frac{1 + \ell}{\ell}\right).$$

Proof of Theorem II: The density d gives rise to a corresponding density d_R on R_E, defined by

$$d_R(A) \equiv d(f^{-1}(A)) \quad \text{for } A \subset R_E.$$

Theorem I applies to d_R.

Since $a_k = o(r^k)$, it is evident that, for any $\varepsilon > 0$,

$$f^{-1}(R_E(x + \varepsilon, y - \varepsilon)) \subset_d E(x, y) \subset_d f^{-1}(R_E(x - \varepsilon, y + \varepsilon)).$$

Hence

$$\log_{10}\left(\frac{y - \varepsilon}{x + \varepsilon}\right) = d_R(R_E(x + \varepsilon, y - \varepsilon) \leqslant d_R(R_E(x - \varepsilon, y + \varepsilon)) = \log_{10}\left(\frac{y + \varepsilon}{x - \varepsilon}\right)$$

and the result follows. Q.E.D.

These results can also be obtained using the measure-theoretic techniques developed in [1]. For a review of the literature on the First Digit Problem, see [5]. It should be noted that the base 10 logarithmic behavior is due to the convention of writing numbers in decimal form. If the numbers were written in base b, then

$$d(R(\ell)) = \log_b\left(\frac{1 + \ell}{\ell}\right).$$

Another example of a density which satisfies (D1) and (D2) is the logarithmic density

$$d_{\log}(A) \equiv \lim_{n\to\infty} \frac{\displaystyle\sum_{k=1}^{n} \frac{\chi(k;\, A)}{k}}{\displaystyle\sum_{k=1}^{n} \frac{1}{k}}.$$

Like the natural density, there exist sets which do not have logarithmic density. The logarithmic density agrees with the natural density wherever the natural density exists, but there are sets which have logarithmic density which do not have natural density. This raises the following questions: Does every density which satisfies (D1) and (D2) agree with the natural density on sets which have natural density? with the logarithmic density? with other summability methods?

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